Notes on

The Evolution of Radiative Transfer Theory

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These notes outline the development of radiative transfer theory (RTT) as it is now used in optical oceanography and ocean color remote sensing. The *sine qua non* of RRT is taken to be the development or use on an equation that predicts the propagation of light through an absorbing and scattering medium. The empirical foundations of RTT are briefly mentioned. The main body of the notes traces, in chronological order of the selected papers, the development of phenomenological radiative transfer equations (RTEs) from the 1880s to the 1940s. Particular emphasis is given to the seminal papers of Lommel (1887) and Ambartsumian (1943). Although most of the early work was driven by astrophysical problems, the resulting equations are applicable to other media such as the oceans. The notes close with comments on the deficiencies of phenomenological RTT and how the recent work of Mishchenko has resolved those deficiencies and connected RTT to fundamental physics. The development of numerical solution techniques for RTEs is not discussed.

Contents

Introduction

Throughout history societies have found various ways—sometimes rather unpleasant—of getting rid of a person when he or she becomes too old to perform any useful work. The modern approach in science is to give someone a lifetime achievement award and then put them to writing history. I seem to be right on schedule, and so far it has been a very enjoyable way to start my dotage.

This informal and brief history of radiative transfer theory (RTT) grew out of an invitation to present an invited talk on that subject at the International Ocean Color Science meeting in Busan, South Korea in April 2019. I am most grateful to the International Ocean Color Coordinating Group and to the Korea Institute of Ocean Science and Technology for enabling me to attend that meeting.

Given the time constraints of an invited talk, I had to choose one path among several through the history of the broad topic of RTT, and I had to choose only a few papers from hundreds and a few persons from dozens for discussion. My goal was to reach the formulation of RTT theory as it is now applied in optical oceanography and ocean-color remote sensing. I therefore consider only papers that present some form of an equation (or data, in the case of Bouguer) that governs the propagation of light through an absorbing and scattering medium. I consider such an equation to be the *sine qua non* of radiative transfer theory. I will, therefore, not discuss the contributions of Rayleigh and Mie because they were interested in scattering by single particles, not in the larger picture of how their work fits into RTT.

As I began to review the literature, the talk naturally organized itself into three periods:

- **Experimental Foundations.** The work of Bougier, Lambert, and Beer.
- **Theoretical Foundations.** The papers of Lommel, Chwolson, Schuster, Planck, Schwarzschild, King, Milne, Gans, Gershun, Ambartsumian, Sobolev, Chandrasekhar, and Preisendorfer.
- **Completing the Bridge.** The recent work of Mishchenko, which clarifies the concepts of radiometry and puts RTT on a firm physical foundation.

There is still another important topic: how to solve the equations once you have them. I have omitted reviewing that history here simply because the development of analytical and numerical solution techniques is an equally large topic, and the time allotted in my talk did not permit their discussion.

Papers before the middle of the 20th century can be rather difficult to understand, even if you can read the requisite Latin, French, German, and Russian. This is because the modern concepts of radiance, plane irradiance, scalar irradiance, intensity, and so on had not yet been formulated in a rigorous fashion, nor was there a distinction between photometry and radiometry. Thus Bouguer's 1729 treatise speaks of "la force de la lumière," which translates literally as "the force

of the light." The German Lommel in 1888 discusses the "die Lichtmenge," which means "the light amount," and "die Leuchtkraft," which means "the illuminating power." Chwolson in 1889 discusses "die Lichtintensität," which is "the light intensity." It was not until Planck (1906) that I find a paper that defines radiance as it is used today. Planck calls his radiance "spezifische Intensität" or "specific intensity," or "Helligkeit" or "brightness". In modern German, radiance is Strahldichte, literally "beam density" or "ray density"; and irradiance is Bestrahlungsstärke, or "radiation density". Neither Strahldichte nor Bestrahlungsstärke ever appear in the early German papers. (Today, in colloquial usage, many Germans just convert the English "radiance" to "Radianz" and never bother with Strahldichte.)

I must also complain that preparing this talk was made no easier by the fact that some journals still give no free access to articles published well over a century ago. For example, the Wiley Online Library wants $38 for a pdf scan of Lommel's 1889 paper originally published in *Annalen der Physik und Chemie.* (The successor journal *Annalen der Physik* is now a part of John Wiley & Sons.) Even though I have access to the Wiley Online Library, they still want $38 after I log in. Thus there are a number of papers I know only by references because they are either simply unobtainable at any price (in particular, the early Russian literature) or because I don't want to spend my lunch money unless I'm sure I actually need the article. Fortunately, I inherited Rudy Preisendorfer's collection of reprints, which does include Lommel (1889) and an English translation of Gershun (1939).

The Google Books Library Project (www.google.com/googlebooks/library/) has scanned many of the classic books, such as Bouguer (1729) and Lambert (1760), as well as some obscure learned-society proceedings such as Lommel (1887), so these can be found on line (untranslated of course). Similarly, Project Gutenberg (www.gutenberg.org) and the Hathitrust Digital Library (www.hathitrust.org/digital_library) have scanned many old and obscure books. These sources were extremely valuable for finding some of the classic papers online. At times I have relied on other reviews for insight into the importance of papers I don't have or can't read.

Well, enough said. The following pages collect some of the information I gathered for presentation during my talk. I allow myself the luxury of including many photos found online, along with biographical tidbits, which although irrelevant to the science may be entertaining. Please let me know if you have other translations for words, or know of other papers I should discuss. This remains a work in progress. Meanwhile, enjoy!

Experimental Foundations

Bouguer. Pierre Bouguer (1698-1758) was a professor of hydrology who wrote on many topics, including award-winning papers on the measurement of magnetic declination at sea, the measurement of the altitude of stars at sea, the design of ship masts, and the first treatise on naval architecture. He is consequently known as "The Father of Naval Architecture." He spent 10 years on a scientific expedition to what is now Equador, the main purpose of which was to measure the length of a degree of latitude at the equator and to determine the exact shape of the earth. This was the first major scientific expedition.

Pierre Bouguer by Jean-Baptiste Perronneau, now in the Louvre (public domain reproduction)

ESSAI
D'OPTIQUE,
SUR
LA GRADATION
DE LA LUMIERE.

Par M. BOUGUER, *Professeur Royal en Hydrographie.*

A PARIS,

Chez CLAUDE JOMBERT, ruë S. Jacques, au coin de la ruë des Mathurins, à l'Image Notre-Dame.

M. DCCXXIX.

Avec Approbation & Privilege du Roi.

In 1729 Bouguer published *Essai d'Optique, sur la gradation de la lumiere* (*Optical Essay on the Dimming of Light*). In 1760 a more complete treatment of the subject, *Traité d'Optique sur la Gradation de la Lumière*, was published posthumously. These works earned Bouguer the sobriquet of "The Father of Photometry," to go along with being the father of naval architecture.

Cover of Bouguer (1729). This book now goes for over $4000 on the rare book market.

Figure 1 shows the method by which Bouguer studied the attenuation of light when passing through a volume of material at B. A candle P illuminates screens D and E. Bouguer at A observed screen D through the material B, and he observed screen E directly. He then changed the distance PE from the candle to screen E until the visual brightness of E matched that of screen D seen through the material. The inverse square law applied to distance PE allowed him to deduce the amount of attenuation of light passing through B. Since the eye is an integral part of the measurement, Bouguer's "la force de la lumière" is what would be called plane illuminance in modern terminology.

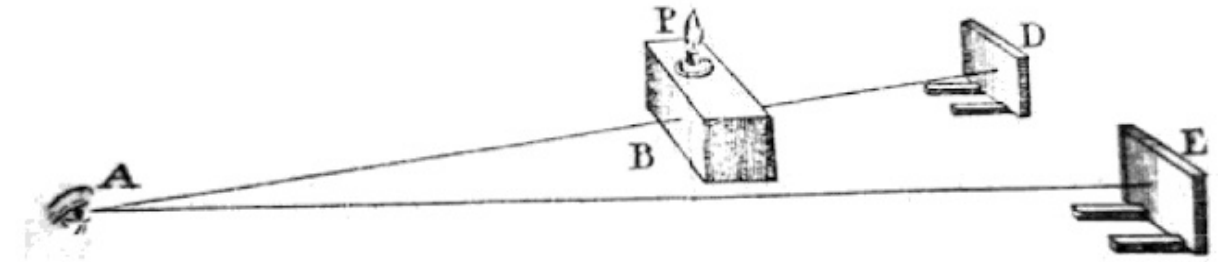


Fig. 1. Bouguer's experimental apparatus (Plate 1, Fig. 5 from Bouguer, 1760).

Using the apparatus of Fig. 1, Bouguer showed that light propagating through a medium decreased on a pattern that could be fit by a logarithmic function. He presented these results in the graphical form seen in Fig. 2.

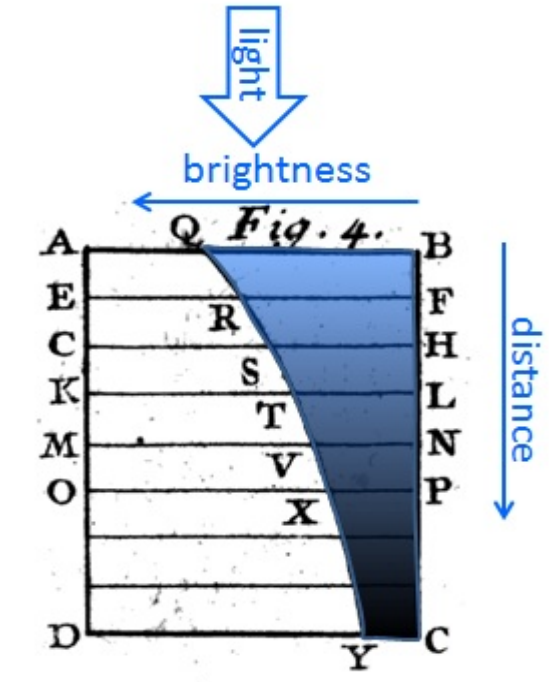
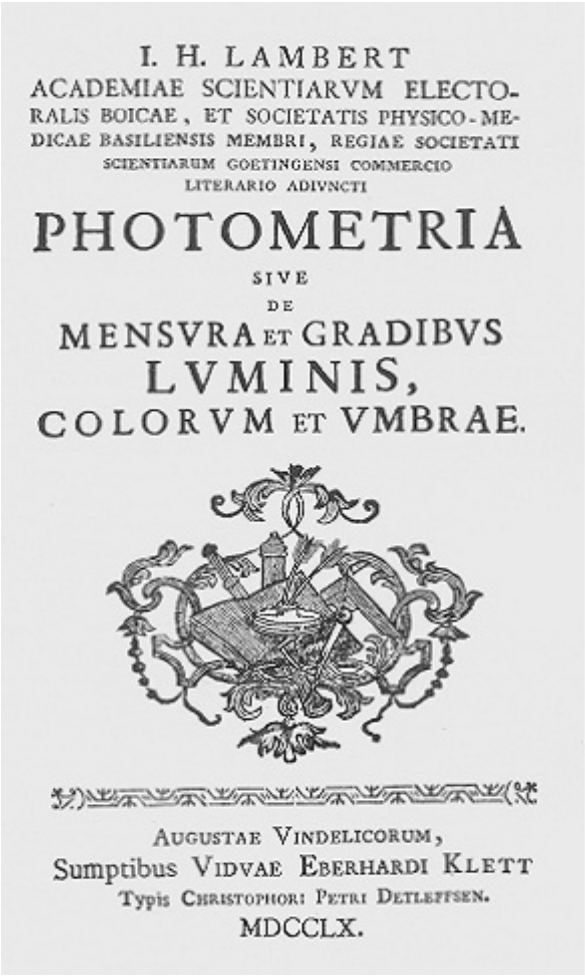


Fig. 2. Figure 4 from Bouguer (1729), enhanced in blue to show the decrease in brightness as light travels further into a medium.

Lambert. Johann Heinrich Lambert (1728-1777) was a Swiss-born prodigy who spent most of his career in Berlin where he found lifelong sponsorship from King Frederick II of Prussia (Oh, the luxury of a tenured hard-money position!). In addition to his work in optics, Lambert made the first proof that π is irrational, and he made a systematic study of map projections (hence the Lambert conformal conic and several other projections that bear his name and are still in use today).

Public domain image from wikipedia.com

In 1760 Lambert published his optics masterpiece, *Photometria, or the Measure and Gradations of Light, Colors, and Shadows*, seen in below in its original Latin. In this work, Lambert cited Bouguer's 1760 *Traité d'Optique* and Bouguer's result seen in Fig. 2.



I. H. LAMBERT
ACADEMIAE SCIENTIARVM ELECTORALIS BOICAE, ET SOCIETATIS PHYSICO-MEDICAE BASILIENSIS MEMBRI, REGIAE SOCIETATI SCIENTIARUM GOETINGENSI COMMERCIO LITERARIO ADIVNCTI

PHOTOMETRIA

SIVE

DE

MENSVRA ET GRADIBVS
LVMINIS,
COLORVM ET VMBRAE.

AUGUSTAE VINDELICORUM,
Sumptibus VIDVAE EBERHARDI KLETT
Typis CHRISTOPHORI PETRI DETLEFFSEN.
MDCCLX.

In *Photometria* Lambert the mathematician put Bouguer's empirical result into mathematical form:

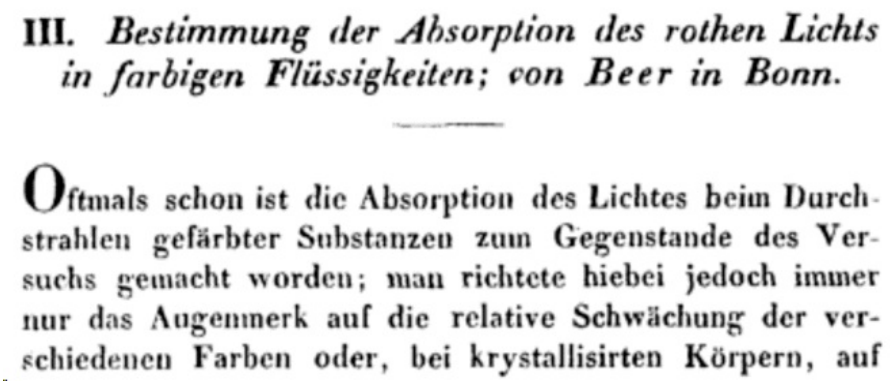
$$I(x) = I(0)\, e^{-Kx}, \qquad (1)$$

where I is some measure of light (e.g., irradiance) and K is an attenuation coefficient (e.g., the absorption coefficient if the medium is non-scattering, or a diffuse attenuation coefficient in general, if I is irradiance). The interpretation of Eq. (1) is that for a given medium (given K), the light decreases exponentially with distance x traveled through the medium.

Lambert also formulated the cosine law for irradiance incident onto a surface: $I(\theta) = I(0)\cos\theta$ where θ is the angle of incidence relative to the normal to the surface. He also studied the reflection by surfaces (hence, Lambertian surfaces) and he introduced the term "albedo."

The cover of Lambert's 1760 *Photometria*. An original of this book sold in 2011 for GBP 27,500 ($36,000) at a Christie's auction.

Beer. August Beer (1825-1863) was a German physicist and chemist. In 1852 he published a paper "Bestimmung der Absorption des rothen Lichts in farbigen Flüssigkeiten" ("Determination of the Absorption of Red Light in Colored Liquids"), in which he investigated the dependence of light transmission on the concentration various salts in aqueous solution. He again found Eq. (1), but now the interpretation is that for a given distance traveled through the medium, the decrease in the light is proportional to the concentration of the absorbing material, which determines the value of K in Eq. (1). This result is still used by chemists to determine concentrations by comparing the transmission of light through a fixed distance of a sample with a known concentration of an absorbing substance to the transmission through the same distance of a sample with an unknown concentration. In that application, K is usually written as the product of a molar absorption coefficient (known) and a molar concentration (to be determined).



III. *Bestimmung der Absorption des rothen Lichts in farbigen Flüssigkeiten; von Beer in Bonn.*

Oftmals schon ist die Absorption des Lichtes beim Durchstrahlen gefärbter Substanzen zum Gegenstande des Versuchs gemacht worden; man richtete hiebei jedoch immer nur das Augenmerk auf die relative Schwächung der verschiedenen Farben oder, bei krystallisirten Körpern, auf

The opening lines of Beer's 1852 paper in *Annalen der Physik und Chemie*.

To summarize, then, it seems that Bouguer should be credited with the original understanding of the exponential decrease of light when traveling through a material medium. Lambert gets credit for putting Bouguer's results into the mathematical form seen in Eq. (1). It's then "Lambert's law" if you use Eq. (1) to predict the attenuation of light as a function of distance for a given substance. It's "Beer's law" if you use Eq. (1) to predict the attenuation of light as a function of concentration for a given distance. To give full credit to everyone, it should be called the "Bouguet-Lambert-Beer law."

If we take the measure of light in Eq. (1) to be a collimated beam of radiance L, for which the attenuation K is the beam attenuation coefficient c, then Eq. (1) is equivalent to

$$\frac{dL(x)}{dx} = -c\,L(x)\,. \tag{2}$$

This is of course the beginning of the radiative transfer equation (RTE) as widely used today. If the medium is non-scattering and without internal sources, then c is the absorption coefficient a and Eq. (2) is the full scalar RTE. We have taken the first step in developing RTT.

Theoretical Foundations

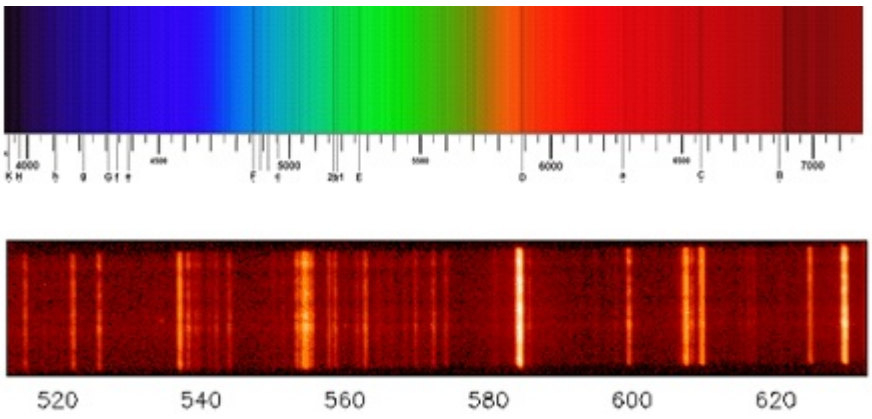
Three Research Topics.

Three different (although related) research topics in astronomy and astrophysics drove the development of radiative transfer theory in the late 1800s and early 1900s.

Topic 1: Predicting the Albedo of Thick Clouds. The first of these problems was the "photometry of diffuse backscatter." By this was meant the problem of predicting the light emitted by an optically thick medium such as a dense cloud given the light incident onto the medium and the physical (absorbing and scattering) properties of the medium. The immediate application was prediction of the albedo of cloud-covered planets such as Venus and Jupiter. This problem was addressed in the papers by Lommel and Chwolson, to be discussed next, and then again by Ambartsumian.

Sulphuric acid clouds on Venus. NASA photo.

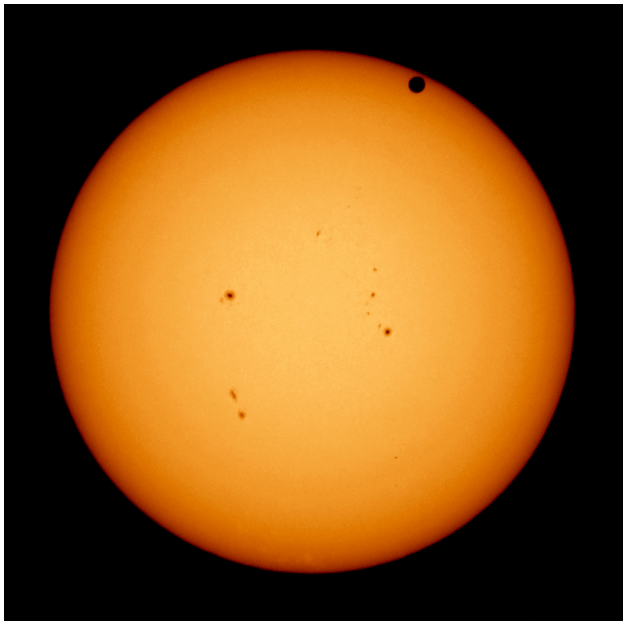
Topic 2: Understanding the Origins of Bright and Dark Lines in Solar Spectra. The second problem was to quantitatively understand the origins of the dark and bright spectral lines in solar spectra. These are illustrated in the figure below:



Solar spectra showing dark lines against a continuum (top) and bright lines (bottom). (Dark line photo: public domain from https://www.flickr.com/photos/140097441@N02/25738920594. Bright line photo from http://www.sr.bham.ac.uk/yr4pasr/project/cds/cds.htm)

It was qualitatively understood that if a relatively cool gas is viewed against a hotter, denser gas generating a continuous blackbody spectrum, then the cooler gas will give dark absorption lines in the continuum. However, if the gas being viewed is hotter than the background, then the gas spectral lines will be brighter than the background. You can observe both patterns in the solar spectrum, depending on exactly what part of the sun is being viewed. The paper by Schuster worked out the quantitative details of how these spectra are generated.

Topic 3: Understanding Solar Limb Darkening. The third "hot topic" of solar physics was to understand the limb darkening seen in photographs of the entire sun, as seen below. Both the brightness of the sun's disk and the color change as the viewing direction goes from the center of the solar disk to the limb (the edge of the disk). This problem was addressed by Schwarzschild and Milne, and has come to be known as "Milne's problem."



Photograph of the sun during the transit of Venus (the dot at upper right) on 05 June 2012. Note that both the brightness and the color change in going from the center to the edge (limb) of the Sun. (Figure from https://commons.wikimedia.org/wiki/File:2012_Transit_of_Venus_from_SF.jpg)

Lommel. Eugen Cornelius Joseph von Lommel (1837-1899) was a German physicist and applied mathematician who spent most of his career at universities in Erlangen and later in Munich. He was an eclectic scholar who read the classics in their original Latin and Greek. In the days when Bavaria still had a king (that lasted until 1918), Lommel was made a Knight First Class of the Order of Merit of the Crown. He wrote on many topics in applied mathematics and optics, including Bessel functions, diffraction, and a number of papers on fluorescence. Interestingly, his obituaries all discuss his work on these topics but make no mention of the paper discussed here, which is his most significant when seen in historical perspective.

Eugen von Lommel (public domain image)

Lommel attacked the problem of diffuse reflectance (the first problem listed above) from a new viewpoint. Earlier researchers had approached this as a problem of predicting the light reflected by the surface of the medium, with a cloud being described by a diffuse reflection function (e.g., a Lambertian BRDF in modern terms), as opposed to specular reflection by a smooth surface (in which case the reflection function is the Fresnel reflectance).

Lommel presented a paper, "Die Photometrie der diffusen Zurückwerfung" ("The Photometry of Diffuse Backscatter"), at the May 7, 1887 meeting of the Mathematical and Physical Section of the Royal Bavarian Academy of Science in Munich. This paper was published in the Academy proceedings for that date and is cited as Lommel, 1887, *Sitzungsber. d. math. phys. Class d. K. B. Acad. zu München*, 17, 95-132. A word-for-word identical version of this paper was later published in 1889 in *Annalen der Physik und Chemie* (*Neue Folge*), 36, 473-502. The subtitle of the 1889 papers says, "From the *Sitzungsber...,zu München*, communicated by Mr. Verf." Perhaps the 1889 paper is "official" version of the 1887 proceedings paper. [You may also see the 1889 paper cited as *Wiedemann's Annalen der Physik und Chemie* because G. H. Wiedemann was editor in chief from 1877-1899.] As we will see, this was a seminal paper for a number of reasons.

Die Photometrie der diffusen Zurückwerfung.

Von E. Lommel.

(Eingelaufen 7. Mai.)

In einer früheren Abhandlung „über Fluorescenz“[1]) habe ich in einem „über die Grundsätze der Photometrie“ überschriebenen Abschnitt gezeigt, dass in der theoretischen Photometrie nicht, wie bis dahin üblich war, die Flächenelemente einer leuchtenden Oberfläche, sondern die Volumenelemente des leuchtenden Körpers als lichtstrahlend zu betrachten seien. Demgemäss wurden der theoretischen Behandlung photometrischer Probleme die folgenden drei Sätze zu Grunde gelegt:

1) Lommel, Wied. Ann. X. 449—472; 631—654. 1880.

The first sentences of Lommel's 1887 Proceedings paper showing the reference to his 1880 paper.

The first sentence of Lommel's 1887 paper (seen above) states (with reference to his 1880 paper, which I do not have) that "... in theoretical photometry it is not, as has been previously assumed, the surface element of a luminous surface that is to be considered, but rather the volume element of the luminous body that is to be regarded as the light emitting element." He then states three fundamental results (or axioms) for photometry:

1. The light amount (Lichemenge) that is traveling from one volume element to another is proportional to the inverse square of the distance.

2. The light amount radiating from a volume element and falling onto a surface element is proportional to the cosine of the incident angle to the surface normal.

3. The light amount radiating from a volume element is decreased along the way by "the absorption law."

The first axiom indicates that Lommel was thinking of "Lichtmenge" as an irradiance: irradiance obeys an inverse square law, but radiance does not.

Lommel's paper is seminal in the development of radiative transfer theory, so I'll spend more time on it than on most other papers. I'll use Lommel's notation, in case you want to compare these notes to his original paper. For the discussion, I'll use the English equivalent of Lommel's terminology. However, after reaching his integral form of the radiative transfer equation, Eq. (6) below, I will comment further on the interpretation of his terminology.

Lommel's paper is the first one I've found that explicitly distinguishes between absorption and scattering, which he quantifies by a "Diffusionsvermögen" ℓ or "diffusion capacity." [In the old papers, "diffusion" is used to mean "scattering" in general, not just diffusion in the modern sense of a process governed by a Fick's law in which a magnitude is proportional to a gradient of a concentration.] Lommel states that this "diffusion capacity" is independent of the incident light and depends only on the properties of the material, and that this quantity is zero for a completely clear body. We can identify this quantity as the scattering coefficient. He assumes that the "diffusion" is equal in all directions (i.e., the scattering is isotropic), in which case the irradiance "decreases with distance by the same law as for absorption." He then introduces an "Absorptionsvermögen" k, which is the modern absorption coefficient. He assumes that k depends on wavelength but that ℓ does not.

Lommel then considers an infinitesimally thin, horizontally infinite, homogeneous layer of material, as shown in my Fig. 3 below. [Unfortunately, papers from Lommel's era rarely contained figures because in those days a figure had to be etched into a copper plate or stone slab for printing. That was no doubt quite expensive, probably even more so than the extra charges for color figures in some journals today.] Referring to Fig. 3, the thin slab is at a depth r' below the surface of the absorbing and scattering medium. Lommel first considers how much light coming from the thin slab is received by a volume element dV located at a distance ρ below the thin slab. Lommel calls F the "Leuchtkraft" ("illumination strength") per unit volume element of the thin slab (dv_{ring} in my figure). Lommel clearly regards F as *emitted* light, so in modern terms, I might call F the luminosity per unit volume. The element of volume in a ring of radius r centered on dV is

$$\mathrm{d}v_{\text{ring}} = r\, \mathrm{d}r\, \mathrm{d}\beta\, \mathrm{d}\rho = (\rho \tan\alpha)\, (\rho \sec^2\alpha\, \mathrm{d}\alpha)\, \mathrm{d}\beta\, \mathrm{d}\rho,$$

where α and β are respectively the polar and azimuthal angles as shown. Integrating over the azimuthal angle β gives the volume of the entire ring element as $2\pi\, \rho^2 \tan\alpha \sec^2\alpha\, \mathrm{d}\alpha\, \mathrm{d}\rho$.

The light quantity (Lichtmenge) received from this ring is then

$$dV \frac{F}{4\pi} \frac{2\pi \rho^2 \tan\alpha \sec^2\alpha\, d\alpha\, d\rho}{(\rho \sec\alpha)^2} e^{-(k+\ell)\rho \sec\alpha} . \qquad (3)$$

In this expression we see the following features:

- The light received is proportional to the receiving volume dV.
- The luminosity of the emitting thin later is isotropic over 4π steradians.
- The light received is proportional to the emitting volume of the ring element.
- The light received is inversely proportional to the square of the distance $\rho \sec\alpha$ from the source element to the receiving element.
- The light is exponentially attenuated by absorption and scattering between the source element and the receiving element.

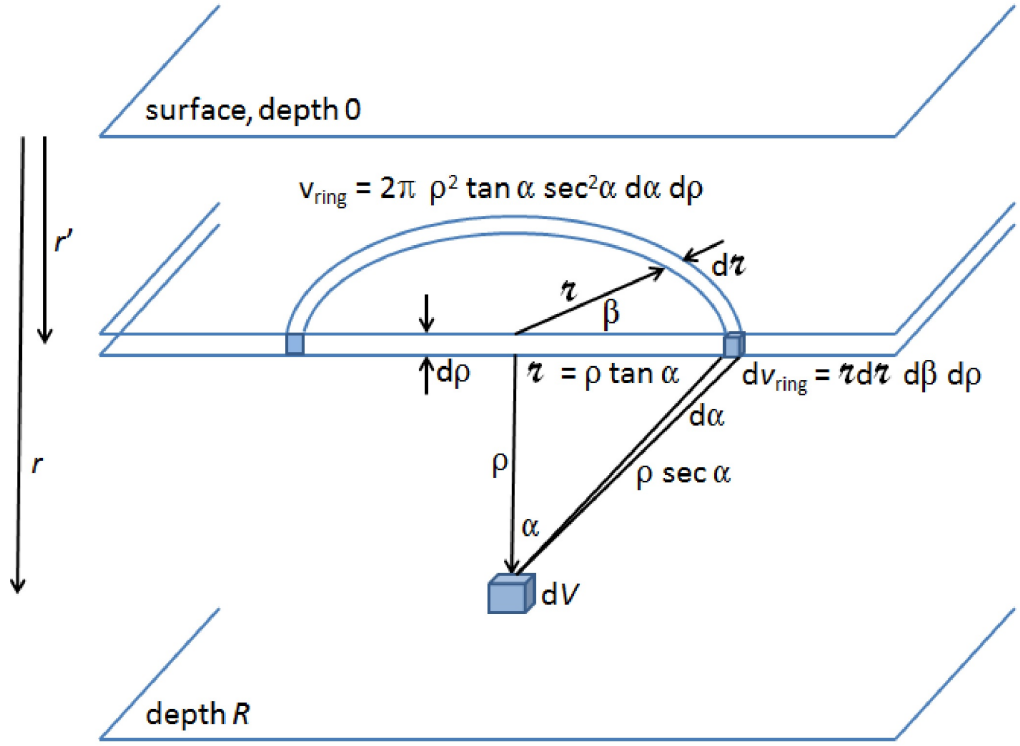


Fig. 3. The geometry of Lommel's derivation.

To get the light received by dV from the entire thin slab, Lommel integrates expression (3) from $\alpha = 0$ to $\alpha = \pi/2$. Letting $m = k + \ell$, this integration involves the factor

$$\int_0^{\pi/2} \tan\alpha\, e^{-m\rho\sec\alpha}\, d\alpha \;=\; \int_\rho^\infty \frac{e^{-mx}}{x}\, dx \;=\; -li(e^{-m\rho})\ ,$$

where $li(x)$ denotes the "logarithmic integral" defined by

$$li(x) \doteq \int_0^x \frac{dt}{\ln t}\ .$$

The light received by dV is then written as

$$-\frac{1}{2} F\, dV\, d\rho\, li(e^{-m\rho})\ .$$

Next, Lommel notes that the luminosity ("Leuchtkraft") F of the thin layer will be a function of the depth r' of the layer below the surface because the light generating F by scattering in the thin layer must first be transmitted from the surface to the depth of the thin layer; thus we have $F(r')$. To get the total light received by dV at depth r, he then integrates the last expression from $r' = 0$ to the bottom of the medium at depth $r' = R$. When the slab is above dV, $\rho = r - r'$ and

$d\rho = -dr'$; when the slab is below dV, $\rho = r' - r$ and $d\rho = dr'$. The total depth integral is then the sum of two parts:

$$f(r) \doteq -\frac{1}{2}\left[\int_0^r F(r')\, li\left(e^{-m(r-r')}\right) dr' + \int_r^R F(r')\, li\left(e^{-m(r'-r)}\right) dr'\right] . \tag{4}$$

Here $f(r)$ is the "total light amount per unit volume" received at depth r from all of the scattering elements of the entire medium from depth 0 to R.

Lommel then notes that the light amount $F(r')$ emitted by the slab at r' comes from two sources. The first is light directly transmitted from the surface to depth r', and the second is light that is scattered into the slab from the rest of the total medium. In the case that the incident light is collimated with an incident angle within the medium of i from the normal to the surface and with a magnitude a, then the directly transmitted term is

$$a\, e^{-(m r'/\cos i)} .$$

The directly transmitted plus scattered light at r′ is then

$$F(r') = \ell\, [\, a\, e^{-m r'/\cos i} + f(r')\,] . \tag{5}$$

The scattering coefficient ℓ in the last equation determines how much of the light reaching the layer at r' is then isotropically scattered into all directions. Inserting terms of the form (5) into the Eq. (4) gives an integral equation for $f(r)$:

$$\begin{aligned} f(r) = -\frac{1}{2}\ell \{ & \int_0^r [\, a\, e^{-m r'/\cos i} + f(r')\,]\, li(e^{-m(r-r')})\, dr' \\ & + \int_r^R [\, a\, e^{-m r'/\cos i} + f(r')\,]\, li(e^{-m(r'-r)})\, dr' \} . \end{aligned} \tag{6}$$

This equation is Lommel's great achievement: an integral form of a radiative transfer equation. He comments that "the function $f(r)$ is of fundamental significance for theoretical photometry." But he also comments that "it does not seem possible, with the mathematical tools now at our command, to obtain a solution (of Eq. 6) in closed form."

Further comment on Lommel's terminology can now be made. He calls a the "light amount of a parallel ray bundle" ("Bezeichnet man...ein paralleles Strahlenbündel ... Lichtmenge mit a"). Equation (5) shows that function $f(r)$ has the same units as a. So if we interpret Lommel's "light amount of a parallel ray bundle" to mean a collimated radiance with units of W m^{-2} sr^{-1}, then Eq. (6) is an equation for (isotropic) radiance. According to Eq. (5), $F(r')$ is then the emitted intensity (W sr^{-1}) per unit volume. If we interpret his "light amount" to mean a collimated plane irradiance with units of W m^{-2}, then function $f(r)$ would also be irradiance, and $F(r')$ would then have units of W m^{-3}, or power emitted per unit volume. In a sense it doesn't matter which

interpretation we give to Lommel's wording because he was assuming that the scattered light, and the resulting light field, are isotropic. In the case of an isotropic radiance distribution, radiance and irradiance differ only by constant numerical factors (with units of steradians): for an isotropic radiance distribution of magnitude L_o, plane irradiance has magnitude πL_o and scalar irradiance has magnitude $4\pi L_o$.

Upon arriving at Eq. (6), we are only 6 pages into a 30 page paper. Immediately after obtaining Eq. (6), Lommel writes

$$f(r) = f_1(r) + f_2(r) + f_3(r) + \cdots$$

where the terms represent higher and higher orders of scattering. He substitutes this expansion into Eq. (6) and obtains a series of equations in which the left hand side involves $f_i(r)$ and the integral terms involve $f_{i-1}(r)$. This is surely the first application of the successive-order-of-scattering solution technique in radiative transfer theory. He computes the single-scattering term $f_1(r)$ analytically, but comments that "...we must to forego computing the higher order terms because the integrals cannot be reduced to known functions."

Lommel then uses his single-scattering analytical solution $f_1(r)$ to examine a number of special cases including very thin slabs ($R \to 0$), very thick slabs ($R \to \infty$), normal incidence, and near-surface $f_1(0)$ and very deep $f_1(\infty)$ behavior. He uses the depth derivative $\mathrm{d}f_1(r)/\mathrm{d}r$ to show that $f_1(r)$ has a subsurface maximum below which $f_1(r)$ decreases monotonically.

He uses $f_1(r)$ to compute the total upwelling light incident onto the surface from the medium below, which he denotes by L. I will omit showing his rather complicated formula for L; but I will comment, based on my teaching experience, that almost no graduate student today could carry out the required calculus. He notes that in the limit of a small value of $\ell/(k + \ell)$ and a large R, his general formula for L depends only on the angle i of the light incident onto the medium, the angle ε of the upwelling light, and the ratio $\ell/(k + \ell)$. In modern terms, the upwelling light from a thick slab depends to first order (the single-scattering solution) on the albedo of single scattering. He summarizes with the comment that these results "...give new formulas for the emission by non-transparent bodies, replacing the cosine law of Lambert."

He next uses his formula for L for the case of normal incidence $i = 0$ to evaluate the quantity

$$M \doteq 2\pi \int_0^{\pi/2} L \sin\varepsilon \, d\varepsilon \ .$$

He calls M "the light amount which streams out from a surface element into all directions of a hemisphere." He says that "for a unit of surface illuminated by a unit light amount, this corresponds to Lambert's concept of 'albedo'." He then evaluates M for the cases of collimated illumination at an arbitrary incident angle i and for uniform illumination of the slab.

For the case of uniform illumination, he gets a result that "corresponds to Seeliger's definition of albedo" citing Seeliger (1886). The simplest of Lommel's reflection formulas

contain the factor

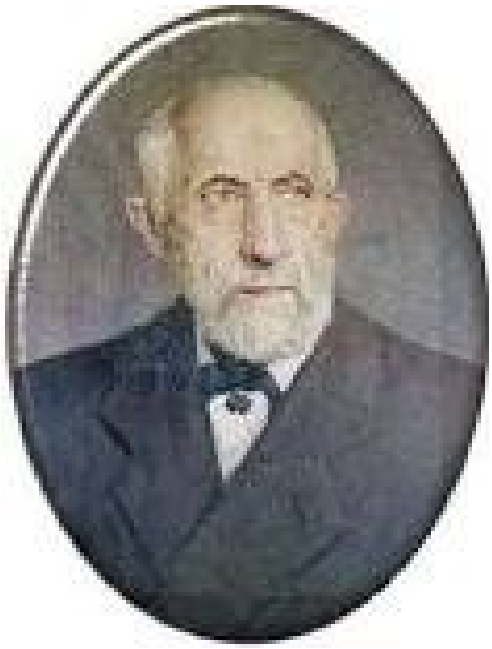
$$\frac{1}{4\pi}\,\frac{\ell}{k+\ell}\,\frac{\cos i \cos\varepsilon}{\cos i + \cos\varepsilon}\,, \tag{7}$$

which Seeliger (1886) attributes to Lommel in his 1880 paper, with the further comment that Lommel "recently" generalized his previous work and cites Lommel's 1887 paper discussed here for the theoretical foundation of the formula. Regardless of the apparent confusion on publication dates, expression (7) is today known as the Lommel-Seeliger reflection law and is still found in astronomy texts. It is solidly based on single-scattering theory and the assumption of isotropic scattering.

To summarize Lommel's contributions:

- He recognized that the albedo of a medium depends on its internal absorption and scattering properties, not just on its surface reflectance properties.
- He made a clear distinction between the effects of absorption and scattering.
- He began with clearly stated axioms and derived an integral form of a radiative transfer equation (RTE) that is still readily recognized as such.
- He recognized that a small volume of material both receives scattered light and is the source of emitted light.
- He recognized the fundamental importance of his RTE.
- He formulated a solution algorithm for his integral equation that is now known as the successive-order-of-scattering technique.
- He used the single-scattering solution of his RTE to evaluate the albedo of a scattering and absorbing medium for a range of approximations.

Chwolson. Orest Danilovich Chwolson (Орест Данилович Хвольсон in Cyrillic) (1852-1934) was a Russian physicist. He is best remembered today as the being the first to predict the possibility of gravitational lensing or the "halo effect" on the appearance of distance galaxies, today called Einstein rings.



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Chwolson is one of those unfortunate Russians who, like Nikita Khrushchev (Хрущёв) is doomed to have his name forever mispronounced by English speakers. The Cyrillic "X" is pronounced something like a gutteral, aspirated English "H", and the "B" has a "V" sound. Then there is the problem of the Russian "hard L" л vs the "soft L" ль. It's hopeless. By the current US Library of Congress transliteration conventions, his name would be spelled Khvol'son, but the closest English sound might be Hvolson. He published the paper discussed below in German, with the spelling Chwolson, so that's what I'll use here. In German the "Ch" would have a "K" sound, and the "W" an English "V" sound, so the Germans of 1889 probably called him Kvolson.

In 1889 Chwolson published a paper titled "Grundzüge einer mathematischen Theorie der inneren Diffusion des Lichtes" (Foundations of a mathematical theory of internal diffusion of light) in the *Bulletin de L'Académie Impériale des Sciences de St. Pétersbourg*. Most papers in that tome are in French, as was the custom for intellectuals in Imperial Russia, but Chwolson's paper was in German. The title has an interesting footnote by the Editor of the Bulletin dated July 4, 1889, which reads in part (my translation)

> The present treatise was submitted to the Academy in the autumn of 1885, but was then withdrawn by the author in order to make another attempt at a complete solution of the main equation, and was then resubmitted almost without change in the autumn of 1888. Because of the appearance in the meantime of a treatise dealing with the same subject by Lommel [referencing Lommel's 1889 paper], the above explanation seemed necessary to me.

In other words, as will be seen below, Chwolson derived in 1885 an integral form of the RTE that is essentially the same as Lommel's Eq. (6) seen above, but Chwolson withheld publishing his result in order to work more on the mathematics of solving the equation. He then resubmitted his paper just after Lommel's work appeared (first in the 1887 Proceedings of the Bavarian Academy of Science, and then in the 1889 *Annalen der Physik und Chemie* paper). Thus, Chwolson may have been the first to derive an integral form of the RTE, but Lommel was first to

publish and now gets the recognition. Lesson learned: if you have something good, don't sit on it.

Chwolson employed the same plane parallel geometry as Lommel and made essentially the same physical and mathematical analysis of how light is absorbed and scattered within a medium. He was specifically interested in the optics of opal glass ("Milchglas" or "milk glass" in German). He starts with a discussion of the angular distribution of light by a single spherical particle in the geometric optics approximation. After considering multiple scatting he argues that it is reasonable to assume isotropic scattering (in modern terminology). He even comments on polarization and argues that it can be neglected because of the depolarization effects of multiple scattering.

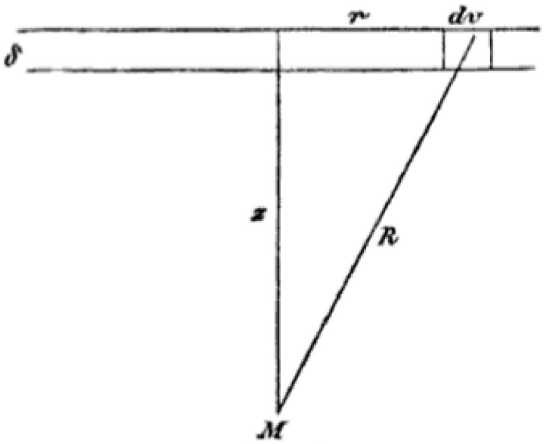

Fig. 2.

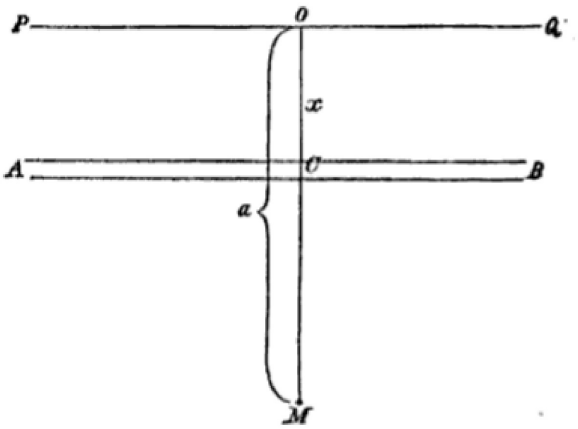

Fig. 3.

Two of Chwolson's figures corresponding to the geometry used by Lommel, but with slightly different notation.

Chwolson eventually arrives at an integral equation for $f(a)$, which he calls the "Lichtintensität," or "light intensity," at depth a below the surface:

$$f(a) = e^{-pa} - \frac{\alpha K}{2}\int_0^a f(x)\,\omega(pa-px)\,dx - \frac{\alpha K}{2}\int_a^h f(x)\,\omega(px-pa)\,dx\,. \qquad (8)$$

Here h is the thickness of the slab and a is the depth of the point of interest within the medium. For light normally incident onto the slab, α is the absorption coefficient due to the particles that are causing the scattering, and p is the sum of the particle and the background-medium absorption coefficients. Chwolson says that it can be assumed that $\alpha = p$ with negligible error in most cases (i.e., the background medium can be assumed non-absorbing). The coefficient $K < 1$ is the albedo of single scattering, which Chwolson calls "the albedo of the particles" ("das Albedo der Theilchen"). Thus α and K are the IOPs under the assumption of isotropic scattering. Finally, $\omega(x) = \mathrm{li}(e^{-x})$ is the same logarithmic integral seen in Lommel's equation.

After obtaining this integral equation, Chwolson says "So far I have not succeeded in calculating the function $f(a)$ for arbitrary values of p, K, and h." In a different place he says, "...a complete solution of the problem does not appear possible...."

Equation (8) as shown above contains the essence of Chwolson's development. However, he also developed a more general version of Eq. (8), which includes a term representing the contribution to $f(a)$ due to the internal reflection of scattered light by the boundaries of the slab. That reflection is described by integrals over direction of the Fresnel reflectance, including total internal reflection at some angles. He points out that the $\exp(-pa)$ term must be replaced by $\exp(-pa/\cos\beta)$ if the incident light is at an angle β to the normal to the slab.

In the remainder of his paper, Chwolson considers various special cases for which Eq. (8) can be solved analytically, at least approximately. In particular, he considers the light field deep within the medium far from the slab boundaries, i.e., at asymptotic depth. His analysis of that problem leads him to exactly the same equation for the asymptotic diffuse attenuation coefficient κ_∞ as is seen on page 469 of *Light and Water* (Chwolson's Eq. 26, with modern notation):

$$1 = \frac{\omega_o}{2\kappa_\infty} \ln\left(\frac{1+\kappa_\infty}{1-\kappa_\infty}\right),$$

which Chwolson says can be solved to obtain κ_∞. He goes on to show that "The same amount of light penetrates to equal optical depths px in all gray materials." His "gray" materials are ones with $K < 1$; a non-absorbing "white" material has $K = 1$ (his K is my ω_o).

Schuster. Franz Arthur Friedrich Schuster (1851-1934) was a German-born physicist who spent most of his career at the University of Manchester in England. He made significant contributions in many areas, including spectroscopy, optics, x-ray imaging for medical diagnosis, and radiative transfer, and he pioneered the use of harmonic analysis (Fourier analysis) to search for "hidden periodicities" in natural phenomena such as weather and earthquakes. He invented the concept of antimatter in two letters to *Nature* in 1898, one of which asked "If there is negative electricity, why not negative gold, as yellow as our own?"

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https://en.wikipedia.org/wiki/Arthur_Schuster

In 1905 Schuster published (in English) his most famous paper, "Radiation through a foggy atmosphere," which addressed the problem dark and bright spectral lines in stellar atmospheres. He starts by saying that "In discussing the transmission of light through a mass of gas, it us usual to consider only the effects of emission and absorption, and to neglect the effects of scattering." He then says, "I call an atmosphere 'foggy' when scattering takes place to an appreciable extent." The problem attacked in this paper was to explain the "dark line" and "bright line" spectra seen in radiating gases such as the sun.

Using simple heuristic arguments and assuming equal amounts of scattering in the forward and backward directions (such as for an isotropic or a Rayleigh phase function), Schuster derived a pair of equations for the energy A traveling in direction $+x$ and the energy B traveling in the opposite $-x$ direction:

$$\frac{dA}{dx} = \kappa(E - A) + \tfrac{1}{2}s(B - A)$$

$$\frac{dB}{dx} = \kappa(B - E) + \tfrac{1}{2}s(B - A)$$

In modern terminology, these are precisely the two-flow equations for E_d (A) and E_u (B) as seen in *Light and Water* Eqs. (5.54) and (5.55), with κ as the diffuse absorption coefficient, $s/2$ as the diffuse backscatter coefficient, and κE as the internal source function, where E is the blackbody emission function. (Schuster assumed that Krichhoff's law holds, namely that in thermal equilibrium the absorptivity of a material equals its emissivity.) In order to have tractable mathematics for an analytical solution of the equations, he restricted himself to the assumption that (in modern terminology) the radiances are hemispherically isotropic, but that the radiances in the upward and downward hemispheres are not of equal magnitude. He recognized that this assumption oversimplifies reality and commented, "The complete investigation leads to

equations of such complexity that a discussion becomes impossible, and I shall only use the solution obtain under the simplified conditions to deduce certain consequences which cannot be affected by the assumption made."

The remainder of the paper then solves the two-flow equations with suitable boundary conditions (no energy incident onto the "top" of the medium; a known amount of energy incident from the "bottom" of the medium). He assumed either a constant source term E, or a linear dependence of E on depth. In his application, E is the temperature-dependent blackbody emission within the stellar atmosphere, so a linear E corresponds to a linear temperature profile (increasing with depth) in the stellar atmosphere. These E functions allowed him to solve the two-flow equations analytically. The result was that the equations can generate either a bright or a dark emission line against the background, depending on the relative importance of the absorption and scattering terms in the two-flow equations and on the strength of the emission within the atmosphere compared to the irradiance incident from below (from deeper within the sun). He showed that scattering is crucial to understanding dark and bright emission lines. The spectrum would be that of a blackbody if scattering is ignored and the medium is optically deep.

The 1905 paper discussed here is often cited as the foundation paper of RTT (by those of are unaware of Lommel and Chwolson) because of its application of the two-flow equations to the quantitative understanding of stellar spectral lines. However, Schuster actually presented the two flows equations themselves in an earlier paper, Schuster (1903; Eq. 24 in that paper). The equations were used to discuss the transfer of heat in the 1903 paper.

Planck. Max Karl Ernst Ludwig Planck (1858-1947) was a German physicist. Planck was a gifted musician and even composed operas. His physics professor in school advised him not to go into physics because "...in this field, almost everything is already discovered,...." Fortunately, Planck ignored this advice and chose physics over music, and the rest is history.

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Planck is best known for his 1900 derivation of the blackbody radiation law (published in 1901), which required the assumption that energy is emitted in discrete quanta. This was the beginning to quantum theory. He received the 1918 Nobel Prize in Physics for that work.

In the winter semester of 1905-06, Planck gave a series of lectures at Berlin University on the theory of heat radiation. He then collected the material of these lectures, along with background material and other results of his work of the previous few years, and published them in book form as *Vorlesungen über die Theorie der Wärmestrahlung* (*Lectures on the Theory of Heat Radiation*; Planck, 1906). The introductory chapter of this treatise is the first place I've found that defines radiance in a thoroughly modern and recognizable form.

The figure below shows part of page 15, where he defines the element of solid angle (in his Eq. 5) and then radiance K (in his Eq. 6). The radiance is defined in the context of the energy that passes through a surface element $d\sigma$ in time dt in a solid angle $d\Omega$ centered on directions (ϑ, φ). Planck calls K the "specific intensity" (spezifische Intensität) or the "brightness" (Helligkeit). The element of solid angle $d\Omega$ is called the "opening angle" (Öffnungswinkel) of a cone. He then defines the plane irradiance (the Gesamtstrahlung or "total radiation") passing through a surface for an aribtrary radiance distribution. Finally, in his Eq. (7), he shows that the "total radiation" is πK if the radiance K is the same in all directions. You don't have to know German to recognize these equations and what they are defining.

The remainder of Planck's 1906 treatise discusses the physics of heat radiation from the standpoint of thermodynamics and Maxwell's equations, rather than via a radiative transfer equation. His contribution to RTT for my present purposes is therefore his formulation of the concept of radiance, or specific intensity as it is still called in some fields. His blackbody formula of course becomes a source term in the RTE if thermal emission is important (as in stellar atmospheres at visible wavelengths, or at thermal IR wavelengths here on Earth).

fortpflanzt. Die Öffnung dieses Kegels ist:

$$d\Omega = \sin\vartheta \cdot d\vartheta \cdot d\varphi . \tag{5}$$

Auf diese Weise erhalten wir für die Energie, welche in der Zeit dt durch das Flächenelement $d\sigma$ in der Richtung des Kegels $d\Omega$ hindurchgestrahlt wird, den Ausdruck:

$$dt\, d\sigma \cos\vartheta\, d\Omega\, K = K \sin\vartheta \cos\vartheta\, d\vartheta\, d\varphi\, d\sigma\, dt . \tag{6}$$

Die endliche Größe K nennen wir die „spezifische Intensität" oder auch die „Helligkeit", $d\Omega$ den „Öffnungswinkel" des von einem Punkte des Elementes $d\sigma$ in der Richtung (ϑ, φ) ausgehenden Strahlenbündels. K ist eine positive Funktion des Ortes, der Zeit und der beiden Winkel ϑ und φ. Die spezifischen Strahlungsintensitäten nach verschiedenen Richtungen sind im allgemeinen gänzlich unabhängig voneinander. Setzt man z. B. in der Funktion K für ϑ den Wert $\pi - \vartheta$, und für φ den Wert $\pi + \varphi$, so erhält man die spezifische Strahlungsintensität in der gerade entgegengesetzten Richtung, eine im allgemeinen von der vorigen ganz verschiedene Größe.

Die Gesamtstrahlung durch das Flächenelement $d\sigma$ nach einer Seite, etwa derjenigen, für welche der Winkel ϑ ein spitzer ist, ergibt sich durch Integration über φ von 0 bis 2π, und über ϑ von 0 bis $\frac{\pi}{2}$:

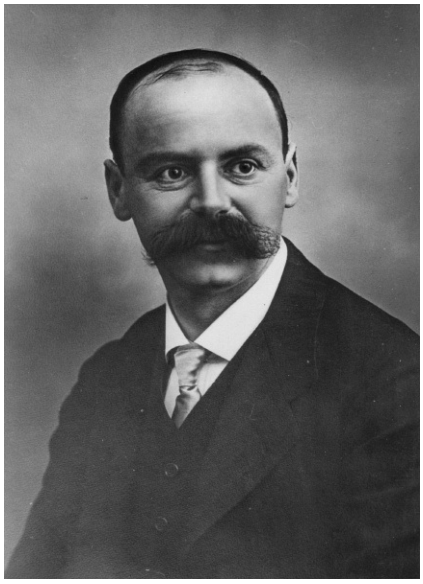
$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta\, K \sin\vartheta \cos\vartheta\, d\sigma\, dt .$$

Ist die Strahlung nach allen Richtungen gleichmäßig, also K konstant, so folgt hieraus für die Gesamtstrahlung durch $d\sigma$ nach einer Seite:

$$\pi K\, d\sigma\, dt . \tag{7}$$

Part of page 15 of Planck (1906) showing his formulation of radiance and its use in computing the plane irradiance.

Schwarzschild. Karl Schwarzschild (1873-1916) was a German physicist and astronomer. He is best remembered today for obtaining the first solution of Einstein's equations of general relativity. That solution yielded the radius of the event horizon of a black hole for a non-rotating sphere, now known as the Schwarzschild radius. Amazingly, he someway found the time to work out that solution in 1915 (as well as to write a second paper on general relativity and one on quantum mechanics in the same year) while serving in the German army as an artillery lieutenant. He died the next year from a rare skin disease that developed while at the Russian front.



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In 1906 Schwarzschild published a paper "Ueber das Gleichgewicht der Sonnenatmosphäre" (On the equilibrium of the solar atmosphere) that began by combining Schuster's two-flow equations with elementary gas thermodynamics and hydrostatics. He used the resulting equations to study the concept of "radiative equilibrium" (Stralungsgleichgewicht), a term he introduced, meaning that the transfer of energy from deeper in the sun to the surface is by thermal radiation, rather than by convection (movement of matter in convective or adiabatic equilibrium, which is the case in the Earth's atmosphere).

He first solved the two-flow equations for temperature lapse rates (the rate of change of temperature with depth in the solar atmosphere) corresponding to both adiabatic (convective) and radiative equilibrium. This analysis showed that the Sun's atmosphere could at least theoretically be in radiative equilibrium.

He then derived a differential equation for the radiance as a function of depth and direction under the assumption that the sun's atmosphere only absorbed and emitted by blackbody radiation; scattering was neglected. This equation (18) and its formal solution for the radiance leaving the solar surface (20) are seen in the figure below:

Strahlung. Man erhält daher für F in völliger Analogie zu (8) die Differentialgleichung:

$$\frac{dF}{dh} = -\frac{\alpha}{\cos i}(E-F) \tag{18}$$

oder:

$$\frac{dF}{d\mu} = -\frac{1}{\cos i}(E-F),$$

wenn man die Abkürzung:

$$\mu = \int_{-\infty}^{h} \alpha\, dh \tag{19}$$

einführt. Die Integration ergiebt für die aus der Atmosphäre austretende Strahlung:

$$F(i) = \int_0^{\infty} E e^{-\mu \sec i}\, d\mu \sec i. \tag{20}$$

Es läßt sich also $F(i)$ berechnen, sobald man die Temperaturverteilung längs der Vertikalen und damit E als Funktion von μ kennt.

Part of page 49 of Schwarzschild (1906). $F(h,i)$ is the upwelling radiance at depth h below the top of the Sun's atmosphere (photosphere) and polar angle i from the normal ($0 \le i \le 90$ deg); α is the absorption coefficient. $E(h)$ is the isotropic blackbody source function, which is a function of the temperature $T(h)$. Schwarzschild's Eq. (18) is exactly what is seen in *Light and Water* Eq. (8.30) for the upwelling radiance in the case of no scattering. Equation (20) gives the radiance leaving the Sun's surface as a function of polar angle i.

Schwarzschild evaluated his Eq. (20) for temperature profiles $T(h)$ (which determine the blackbody source term E as a function of depth in the solar atmosphere) corresponding to either convective or radiative equilibrium of the Sun's outer atmosphere. These evaluations give an angular dependence of the radiance (normalized to 1 at the Sun's center) leaving the top of the Sun's atmosphere to be

$$F(i) = \frac{1+2\cos i}{3} \quad \text{for radiative equilibrium (Schwarz. Eq. 28)}$$
$$= (\cos i)^{4(\gamma-1)/\gamma} \quad \text{for convective equilibrium (Schwarz. Eq. 29)},$$

where i is the angle from the normal, and γ is the "head capacity ratio" or "insentropic index". The constant γ depends on the degrees of freedom in the gas molecule: γ= 5/3 for a monoatomic gas, γ = 7/5 for a diatomic gas, and so on.

His paper closes with the table seen below, which compares the predictions of his Eqs. (28) and (29) with measured radiances across the sun's diameter. This comparison supports his hypothesis that the Sun's photosphere is in radiative equilibrium.

	$\frac{r}{R}$	Messung. (Measured)	Strahlungs-gleichgew. (Radiative equilibrium)	Adiabat. Gleichgew. (Convective equilibrium)
Sun's center	0.0	1.00	1.00	1.00
	0.2	0.99	0.99	0.98
	0.4	0.97	0.95	0.92
	0.6	0.92	0.87	0.80
	0.7	0.87	0.81	0.71
	0.8	0.81	0.73	0.60
	0.9	0.70	0.63	0.44
	0.96	0.59	0.52	0.28
	0.98	0.49	0.47	0.20
Sun's limb	1.00	(0.40)	0.33	0.00

Schwarzschild's table (annotated) of normalized radiances across the Sun's image as computed for temperature profiles corresponding to radiative and convective equilibrium, compared with measurements.

Schwarzschild recognized the crudeness of the assumptions in his 1906 paper and stated in the introduction that "The whole analysis can by no means be considered conclusive or compelling, but it may lead to further speculation by first expressing a simple thought in the simplest form." There is no better way to state the value of a first-order model.

In 1914 Schwarzschild returned to this topic in a second paper, "Über Diffusion und Absorption in der Sonnenatmosphäre" (Scattering and absorption in the Sun's atmosphere). In this paper he worked with radiance "...in a manner analogous to that of Planck" (1906). He developed a pair of RTEs for upwelling and downwelling radiances, including the contribution of isotropic scattering. These equations are exactly the pair of equations seen in *Light and Water* Eq (8.30), for the case of isotropic scattering.

He then used equations to simulate the changes observed in two of the Fraunhofer lines of calcium in the solar spectrum as the viewing direction moves from the center of the Sun to the limb. In order to solve the equations analytically, he considered only two extreme cases: that of absorption and emission (as in his 1906 paper), and that of only scattering (i.e., with no absorption or emission within the layer being viewed; the blackbody radiation enters the layer at the lower boundary). He found that the solution for only scattering gives a better agreement with observation than the assumption of absorption and emission without scattering. He also evaluated the numerical magnitude of the errors on Schuster's 1905 paper resulting from Schuster's assumption that the radiances were isotropic in the upward and downward hemispheres.

King. Louis Vessot King (1886-1956) was a physics professor at McGill University in Montreal. His obituary (Foster, 1957) called him a "Gifted inventor and foremost mathematical physicist in Canadian history" and an "immortal among Canadian scientists." Much of his research was on submarine acoustics (during WWI) and on heat transfer. He invented the hot wire anemometer.

Photo from Foster (1957)

In 1913 King published a long paper (King 1913a, 1913b) "On the Scattering and Absorption of Light in Gaseous Media, with Applications to the Intensity of Sky Radiation." He begins by noting that "Each element of volume will scatter a certain proportion of the radiation incident upon it, so that each element besides being illuminated by the incident radiation is also subject to the aggregate radiation from all the other elements within the surface Σ, i.e., to the effect of *self-illumination*." (italics his). He then derived an integral equation for the radiance:

obtain the following *integral equation* for the scattered radiation at and from any point,

$$\mathrm{I}(x, y, z, 0, \theta) = \mu(\theta)\,\mathrm{E}(x, y, z) + \int_{\Sigma} \mu(\widehat{rr'})\,\mathrm{I}(x', y', z', 0, \theta')\, r'^{-2}\, e^{-\int_0^{r'} \mathrm{K}\, dr'}\, dv', \quad . \quad (14)$$

In this equation, radiance $I(x,y,z,r,\theta)$ is the radiance in direction θ at distance r from the point of interest at (x,y,z). E is the irradiance reaching point (x,y,z), and $\mu(\theta)$ is a function that scatters the irradiance E into radiance I in direction θ. His K in the exponential is clearly described as the sum of absorption and scattering coefficients (in modern terminology). His $\mu(\widehat{rr'})$ factor is the volume scattering function (VSF). His Eq. (14) is notable in that the VSF is arbitrary, so this is the first formulation of an RTE that does not begin by assuming isotropic scattering, or something similar. King then compares $\mu(\widehat{rr'})$ with the Rayleigh VSF, which is proportional to $(1 + \cos\theta)$, and later takes $\mu(\widehat{rr'})$ to be isotropic for numerical calculations.

King the mathematician notes that his Eq. (14) "is of the Fredholm type", which in general has the form

$$u(x) = f(x) + \int_{x_1}^{x_2} u(\xi)\, K(x,\xi)\, d\xi\,.$$

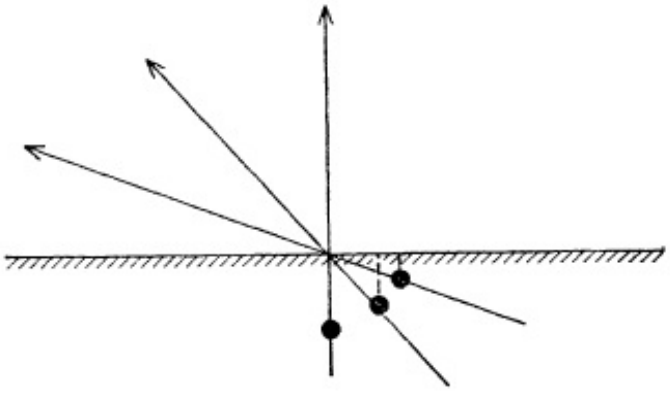
He then comments that "except for special forms of the kernel $K(x,\xi)$," such equations "do not lend themselves easily to numerical evaluation." He then goes on to develop approximate solutions of his Eq. (14) under various assumptions to obtain an approximate formula for the radiance of the sky viewed in any direction. After 15 pages of math, he notes that his solution for the sky radiance contains terms that "represent the contribution of self-illumination to the scattered radiation coming from any particular direction. An evaluation of this effect has not, so far as the writer is aware, been submitted to calculation although the importance of the effect is realized both by Kelvin and Rayleigh, and in an analogous problem by Lommel." A footnote then references Lommel (1887) and notes that Lommel's problem "...is included as a particular case of the investigation of the present paper." This is the last reference I have found to Lommel until 1980.

Milne. Edward Arthur Milne (1896-1950) scored the highest score ever made (up to that time, at least) on his entrance exam in math and natural science at Cambridge Univ in 1914. He became a distinguished mathematician and one of the founders of modern theoretical astrophysics.

In the 1920s, he focused on the thermodynamics and radiative properties of stellar atmospheres. "Milne's Problem," as it is now known, is to find the radiance leaving the surface of a semi-infinite, plane-parallel, isotropically scattering atmosphere that is in radiative and thermodynamic equilibrium. Solving this problem is the key to understanding limb darkening of the sun, which was the third research topic listed above. This is the same problem first addressed in Schwarzschild's 1906 and 1914 papers.

Photo from
https://www.encyclopedia.com/science/dictionaries-thesauruses-pictures-and-press-releases/milne-edward-arthur

In 1921 Milne published a classic paper on the problem of limb darkening. He recognized that when viewing the sun a different angles from the local normal, the radiance comes predominately from different depths within the solar atmosphere and is attenuated along different slant paths through the atmosphere, as illustrated below. Different depths within the solar atmosphere are at different temperatures (with temperature increasing going into the sun), so the blackbody radiation will shift to the red for the cooler, near-surface temperatures seen at near-grazing angles.



Geometry of Milne's analysis. The dots represent one optical attenuation length from the surface as seen along various paths. (Figure from Chandrasekhar (1980).)

Milne assumes a plane parallel solar atmosphere, with depth x being measured inward into the sun from $x = 0$ at the solar surface. Polar angle θ is measured from the $-x$ direction, so that at $x = 0$ polar angle $0 \le \theta \le \pi/2$ is radiance leaving the Sun's surface, and $\pi/2 < \theta \le \pi$ is radiance incident onto the Sun's surface. The opening section of this paper then shows the following:

absorption, supposed independent of wave-length. Lastly, let B (a function of x) be the intensity of black-body radiation corresponding to the temperature of the matter at any point x. By Kirchoff's law the emission per unit volume in all directions at any point x is $4\pi k\rho B$. By considering in the usual way the gains and losses of a narrow pencil of radiation during a short stretch of its paths, it is found that I must satisfy the equation

$$\cos\theta \frac{dI}{dx} = k\rho(I - B) \quad . \quad . \quad . \quad . \quad (1)$$

Setting

$$\tau = \int_0^x k\rho dx,$$

equation (1) becomes

$$\cos\theta \frac{dI}{d\tau} = I - B \quad . \quad . \quad . \quad . \quad (2)$$

So, by 1921 the RTE in a recognizable, modern form was well enough known that Milne felt no need to derive it (as did Schwarzschild), but could just state that "...considering in the usual way the gains and losses..." gives the equation with which he will work.

For completeness of these notes, it is worthwhile to outline Milne's development of his Eqs. (1) and (2). In Eq. (1), ρ is the density [kg/m^3], and k is the "coefficient of mass-absorption". According to Eq. (1), $k\rho$ must have units of the modern absorption coefficient [1/m], so we recognize k as a mass-specific absorption cross-section with units of m^2/kg. Milne assumed that k was independent of wavelength, and he called I "the intensity of radiation of all wave-lengths", so he is thinking of I as broad-band radiance with units of W/(m^2 sr), not spectral radiance or specific intensity. The blackbody source term clearly B has the same units as I. Kirchoff's radiation law states that in thermodynamic equilibrium, the rate of energy absorption must equal the rate of energy emission into all directions at a point x. The rate of absorption of radiant energy [Joules per second or Watts] per unit volume is

$$k\rho \int_0^{2\pi}\int_0^{\pi} I \sin\theta \, d\theta \, d\varphi = 2\pi k\rho \int_0^{\pi} I \sin\theta \, d\theta \, d\varphi \qquad \left[\frac{W}{m^3}\right]$$

where the second equation results from the assumption that the radiance distribution I is azimuthally isotropic. The rate of emission of isotropic blackbody radiation into all directions [4π sr] is $4\pi k\rho B$. Equating these expressions for the rate of absorption and the rate of emission gives the condition for radiative equilibrium:

$$2B = \int_0^{\pi} I \sin\theta \, d\theta \, . \qquad \text{(Milne Eq 3)}$$

Milne then states that the problem is to solve Eq. (1) subject to the requirement of Eq. (3) and the boundary condition $I(x = 0, \theta) = 0$, for $\pi/2 < \theta \le \pi$, i.e. the boundary condition that there is no incident radiance at the solar surface.

After taking into account the sign change of $\cos\theta$ for upward ($\cos\theta > 0$) and downward ($\cos\theta < 0$) radiances, Milne writes the "formal solution" of Eq. (2) as

$$I(\tau,\theta) = e^{\tau\sec\theta} \int_{\tau}^{\infty} B(t) \sec\theta \, e^{-t\sec\theta} \, dt \qquad \text{for } 0 \le \theta \le \pi/2 \qquad \text{(Milne Eq. 17)}$$

$$I(\tau,\psi) = e^{-\tau\sec\psi} \int_{0}^{\tau} B(t) \sec\psi \, e^{-t\sec\psi} \, dt \qquad \text{for } \pi/2 < \theta \le \pi \text{ or } 0 < \psi \le \pi/2 \qquad \text{(Milne Eq. 18)}$$

Of course, these equations do not immediately give the radiances because the depth distribution of the blackbody source term is unknown until the depth distribution of temperature within the solar atmosphere is specified (the blackbody emission depends only on the temperature). Note, however, that Eq. 18 evaluated at $\tau = 0$ gives 0, which satisfies the boundary condition of zero incident radiance. Equation (17) shows that the radiance leaving the surface at $\tau = 0$ depends on the depth profile of the temperature $T(\tau)$ via $B(T(\tau))$ weighted by an exponential function of depth and direction. Milne then inserts his formal solutions for I, Eqs. (17) and (18), into the condition for radiative equilibrium, his Eq. 3. *This gives an integral equation for the depth distribution of* B:

$$2B(\tau) = \int_0^{\pi/2} e^{\tau\sec\theta} \sin\theta \, d\theta \int_{\tau}^{\infty} B(t) \sec\theta \, e^{-t\sec\theta} \, dt + \int_0^{\pi/2} e^{-\tau\sec\psi} \sin\psi \, d\psi \int_{0}^{\tau} B(t) \sec\psi \, e^{-t\sec\psi} \, dt \qquad \text{(Milne Eq. 19)}$$

After a good bit of calculus, he finally transforms this equation into

$$B(\tau) = \frac{1}{2} \int_0^{\infty} B(t) \, Ei(|t-\tau|) \, dt \qquad \text{(Milne Eq. 47)}$$

where Ei is the exponential integral defined by

$$Ei(z) = \int_z^{\infty} \frac{e^{-y}}{y} \, dy \, .$$

Equation (47), now called "Milne's equation" allows the blackbody function, hence the temperature, to be determined as a function of depth within the solar atmosphere, for the assumptions of the derivation. Once $B(\tau)$ is known, this source function can be inserted into Milne's Eq. (17), evaluated at $\tau = 0$, to determine the radiance distribution leaving the Sun's surface. This $I(0,\theta)$ can then be compared with measurements of the solar limb darkening, as was done by Schwarzschild.

Of course, in the pre-computer days of a century ago, Milne's Eq. (47) had to be solved by pencil and paper. Milne developed a method of successive approximations starting with a first approximation of the form $B(\tau) = a + 2b\tau$, where a and b are constants to be determined by the requirements of radiative equilibrium and the total flux emitted by the Sun. He compared his solution with those of Schwarzschild and showed that the Milne prediction of $I(0,\theta)$ gave a somewhat better agreement with observation than did the models of Schwarzschild (or of Eddington and Jeans, who were also working on this problem). The importance of Milne's equation in astrophysics is so great that it spawned a cottage industry developing mathematical ways to solve the equation (e.g., Hopf, 1934), and papers are still being published today on numerical techniques for solving Eq. (47).

It must be remembered that Milne's Eq. (47) was derived for a semi-infinite, homogeneous, plane-parallel medium with isotropic scattering. That will take you a long way in studies of stellar atmospheres, but no where at all in studies of the ocean, where phase functions are highly peaked in the forward direction. That's why you don't see equations like this in *Light and Water*. Nevertheless, this equation was a major advance in RTT.

Gans. Richard Martin Gans (1880-1954) was a German-born and -educated physicist who spent most of his scientific career at universities in Argentina. He is best remembered today for his solutions of Maxwell's equations for scattering by prolate and oblate spheroids (corresponding to Mie's solution for spherical particles) and for the Rayleigh-Gans approximation for scattering by "optically soft" particles.

Photo from Swinne (2017)

In 1924 Gans published a paper "Die Farbe des Meeres" (The Color of the Sea). This is the first radiative transfer paper I've found that refers to the sea. Gans opens by commenting on C. V. Raman's correction of Lord Rayleigh's infamous statement that "The much-admired dark blue of the deep sea has nothing to do with the colour of water, but is simply the blue of the sky seen by reflection."

1924. № 17.

ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 75.

1. ***Die Farbe des Meeres;***
von R. Gans.

Der Meinung Lord Rayleighs[1]), daß die dunkelblaue Farbe der tiefen See sich einfach durch Reflektion des blauen Himmels erklärt, stellte Raman[2]) die fraglos richtige Behauptung gegenüber, daß die Erscheinung durch molekulare Zerstreuung des Sonnenlichts (Tyndalleffekt) an den Wasserteilchen hervorgerufen wird, wobei aber die wahre Absorption des Wassers wesentlich mitspielt. Diese Gedanken hat Ramanathan[3]) auf Anregung Ramans weiter verfolgt.

The first paragraph of Gans' paper on the color of the sea.

Gans then says he wants to take a more thorough look at the problem than has been done previously, namely to account for radiance incident onto the sea surface at an arbitrary angle to the zenith, radiance exiting the sea surface at any angle and, most importantly, to account for polarization. Unlike any previous paper discussed here, Gans worked entirely with electric fields in his very mathematical paper. This is not surprising, given his expertise at solving Maxwell's equations. He resolves the state of linear polarization into components parallel and perpendicular to the meridional planes of the incident and scattered radiances. To obtain irradiance, he evaluates the Poynting vector using his computed electric fields.

Gans first obtained a solution for the electric fields of the light within and leaving the water in the single-scattering approximation. For example, his equations for the components of the polarized specific intensity (radiance) K perpendicular (subscript s, for senkrecht) and parallel (subscript p) to the final meridional plane, for the case of incident light polarized parallel to the incident meridional plane (Fall p, meaning Case p) are

ist, so ergibt sich für die spezifische Intensität K in Luft:

$$(22)\quad \begin{cases} K_s = \dfrac{3}{16\pi}\,\dfrac{h}{h+h'}\,\dfrac{J_0\, d_p(\alpha)\, d_s(\Theta)}{n^2}\,\dfrac{\cos\beta\cos^2\vartheta\sin^2\varphi}{\cos\beta+\cos\vartheta}, \\ K_p = \dfrac{3}{16\pi}\,\dfrac{h}{h+h'}\,\dfrac{J_0\, d_p(\alpha)\, d_p(\Theta)}{n^2}\,\dfrac{\cos\beta\cos^2\varphi}{\cos\beta+\cos\vartheta}. \end{cases} \qquad \text{(Fall } p\text{)}$$

A similar pair of equations gives the radiance components for the case of the incident light polarized perpendicular to the incident meridional plane. In these equations, J_o is the magnitude of the incident radiance, $d_s(\Theta)$ and $d_p(\Theta)$ are functions of the incident and final angles, and n is the index of refraction of the water. These equations involve a factor $h/(h + h')$, where Gans calls h' "true absorption" and h "the coefficient of apparent absorption generated by scattering of the light" (i.e., the scattering coefficient); he calls $h + h'$ the extinction coefficient. His ratio $h/(h + h')$ is the albedo of single scattering ω_o in modern terminology.

Gans then notes that, for a given spectrum of the incident radiance J_o, the wavelength dependence of the water-leaving radiance determined by the $h/(h + h')$ factor, except for a weak wavelength dependence in the index of refraction n. He then asks how the ocean would look "...if the water had no true absorption ($h' = 0$), but only extinction as a consequence of light scattering. Then $h/(h + h')$ would equal 1, and the wavelength dependence would disappear from our formulas, except for the inconsequential dispersion in n. The backscattered light would therefore be white." (if J_o is a white light spectrum). He then concludes (italics his), "We can therefore say: *That the ocean sends out diffuse light at all is due to molecular light scattering; that this light is blue is explained by the true absorption in the red, yellow, and blue.*"

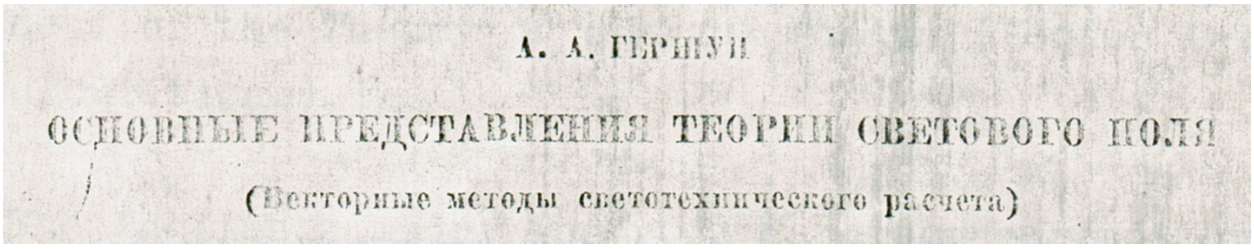
Gans then acknowledges that his single-scattering solution is valid only for small values of $h/(h + h')$. He then develops a "rigorous formulation of the problem." After many pages of math, he eventually arrives at a pair of coupled, inhomogeneous, linear, differential-integral equations for the parallel and perpendicular components of the radiance. These equations, along

with suitable boundary conditions at the sea surface and bottom (at depth ∞), determine the in-water radiance distribution. He then devotes another 8 pages to approximate solutions of these equation using power series expansions in the parameter $h/(h + h')$. The math gets progressively uglier, with little gained for the physics, which was extracted from the single-scattering solution.

Gershun. Andrei Aleksandrovich Gershun (Russian: Андрей Александрович Гершун) (1903-1952) was a Russian physicist who did much work in photometry, colorimetry, and optics. He is regarded both as the father of what might be called "lighting technology" and as the father of Soviet optical oceanography. Although he published over 90 papers and books, he is known in the west for Gershun's law and for the Gershun tube radiometer.

Photo from https://ru.wikipedia.org/wiki/Гершун,_Андрей_Александрович

In 1936 Gershun published a paper, Основные представления теории светового поля (векторные методы светотехнического расчета) (Fundamental Ideas of the Theory of Light Fields (Vector Methods of Light Calculations)) (Gershun 1936a). The title lines of this paper are seen below.



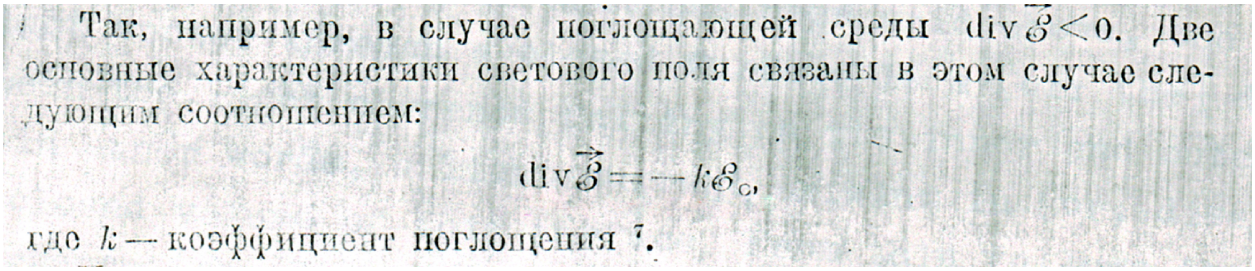
А. А. ГЕРШУН

ОСНОВНЫЕ ПРЕДСТАВЛЕНИЯ ТЕОРИИ СВЕТОВОГО ПОЛЯ

(Векторные методы светотехнического расчета)

The heading of Gershun's 1936 Izvestia paper. Preisendorfer's files contained only two nearly illegible pages of this paper: the cover page and the page with Gershun's equation, seen below. I have been unable to find the full version of the original paper, which as far as I know has never been translated into English.

The figure below shows the page of the 1936 Izvestia paper showing the divergence of the vector irradiance equal to the the negative of the absorption coefficient times the scalar irradiance: this result is now known as Gershun's equation or Gershun's law.



Так, например, в случае поглощающей среды $\operatorname{div}\vec{\mathcal{E}} < 0$. Две основные характеристики светового поля связаны в этом случае следующим соотношением:

$$\operatorname{div}\vec{\mathcal{E}} = -k\mathcal{E}_0,$$

где k — коэффициент поглощения [7].

The presentation of Gershun's equation in the 1936 Izvestia paper.

Gershun also published a book in 1936, световое поля (*The Light Field)* (Gershun 1936b). The importance of that book was apparently immediately recognized because it was translated into English and published in the U.S. in 1939 as *The Light Field* (Gershun, 1939). Gershun's goal in this book was to formulate a physical and mathematical treatment of the light field in the same manner as is used for electric or gravitational fields. He thus carefully works his way through definitions of the fundamental physical quantities and develops mathematical relations among those quantities using the same vector calculus that is applied to other vector and scalar fields. His development has a level of logical and mathematical rigor that is similar to the much later works by Preisendorfer. Gershun also discusses ways to measure the radiometric quantities he defines—he is, after all, the inventor of the Gershun tube radiometer.

Gershun defines a scalar quantity that he calls the "space illumination;" this is the scalar irradiance in modern terminology. Gershun's "light vector" is what is now called the vector irradiance (symbol $\vec{\mathcal{E}}$ in the equation above, $\boldsymbol{D}$ in the equation below). He states that "These two quantities are the fundamental functions of position in the light field." Only after 103 pages of development does he arrive at the relation between the space illumination and the light vector—namely his eponymous equation. In his book he defines a quantity *f* as "the space density of light, produced per unit time" and notes that "If we deal with absorption rather than emission, the values of *f* will be negative." He then presents the equation

$$f = \operatorname{div} \boldsymbol{D} \qquad \text{(Gershun Eq. 65)}$$

and states "*The space density of light, produced (or absorbed) per unit time, is equal to the divergence of the light vector*." (his italics) This is the form of the divergence equation in the book. It includes the form shown in the Izvestia paper if absorption is the physical process changing the space illumination, in which case $f = -k\mathcal{E}_o$.

Gershuns's book is of historical significance for the development of RTT because *it was the first serious attempt to place radiometry on a firm physical foundation. That is to say, its goal was to make a previously phenomenological theory a part of physics.* (A phenomenological

theory mathematically describes observed physical phenomena without connecting the description to fundamental physics. A phenomenological theory is usually developed using intuition based on observation and heuristic arguments. The resulting equations may give good results, but sometimes not good understanding.) This is a topic to which we will return in the last section of these notes.

Gershun published a number of other papers on optical oceanography, e.g. Berezkin, Gershun, and Yanishevskiy (1940), “Transparency and color of the Sea." I am unable to find these papers, but they would no doubt be of historical interest.

Ambartsumian. Victor Amazaspovich Ambartsumian (Russian: Виктор Амазаспович Амбарцумян) (1908-1996) was an Armenian/Soviet astrophysicist who worked in many areas. He is internationally recognized as one of the founders of modern astrophysics. He is today a national hero in Armenia, and his picture is on the Armenian 100 dram note, seen here. The Armenian government today awards a biennial Victor Ambartsumian International Prize, which generally goes to someone who has greatly advanced astrophysics; the prize is currently worth US$300,000.

Ambartsumian spent the 1930s at Leningrad University working on astrophysical problems. His amazing memoir (http://vambartsumian.org/) tells how he was awakened at 4 AM on June 22, 1941 by the sound of German bombers flying over Leningrad, when Germany opened its surprise attack on the USSR. He was soon ordered to evacuate Leningrad, taking as much equipment and scientific personnel as possible with him. On July 17 he and his family left Leningrad in a railroad freight car, heading eastward. It took five days just to reach Moscow, where he endured another night of German bombing. He eventually reached a small town in what is today Tatarstan, where he spent the next four years working on problems of military importance. It is to me almost incomprehensible, but even in the chaos of war and burdened by his wartime duties, Ambartsumian someway found the time and mental discipline to publish seminal papers in 1942, 1943, and 1944, which have had a major and long-lasting impact on RTT.

These three papers revisited the problem of reflectance by optically thick atmospheres. This was the same problem addressed by Lommel and Chwolson over 50 years before. However, Ambartsumian developed an entirely new approach. The 1943 paper, "On the problem of the diffuse reflection of light by a turbid medium," was particularly elegant. It is only 6 pages, but the development therein contains the seed of what today is the mathematical core of HydroLight. Given that I have spent much of my own career solving the RTE using a technique that traces back to this paper, I will discuss it in some detail.

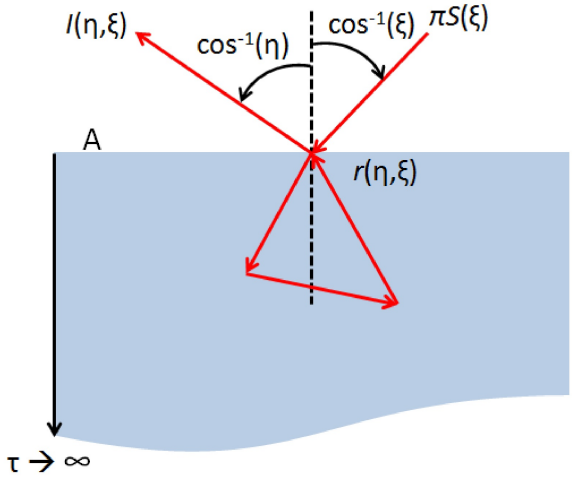
[Note: In oceanography, we use E for the plane irradiance. Astrophysicists often use πS for the same quantity. This was apparently done so that for an isotropic light field of radiance L_o, the

plane irradiance would have the same magnitude as the radiance (but different units of course). For an isotropic radiance, you will see $E = \pi L_o$ in *Light and Water*, but $\pi S = \pi L_o$, i.e., $S = L_o$, in much of the astrophysical literature (e.g., Chandrasekhar, 1950, calls πF the flux; Ambartsumian uses πS). One astrophysics text I consulted says the π is there "for historical reasons" and that πS is called the "astrophysical flux." (The astrophysicists also have an "Eddington flux" and a bunch of other stuff that isn't used in oceanography.) It's just a matter of definitions, but it's confusing and a good way to lose factors of π when going from one field to another.]

The problem is to compute the radiance reflectance of an optically infinitely thick, homogeneous, plane parallel, absorbing and scattering medium, given the inherent optical properties of the medium and the incident irradiance. Ambartsumian defines this reflectance as

$$r(\eta,\xi) = \frac{I(\eta,\xi)}{S(\xi)}, \tag{9}$$

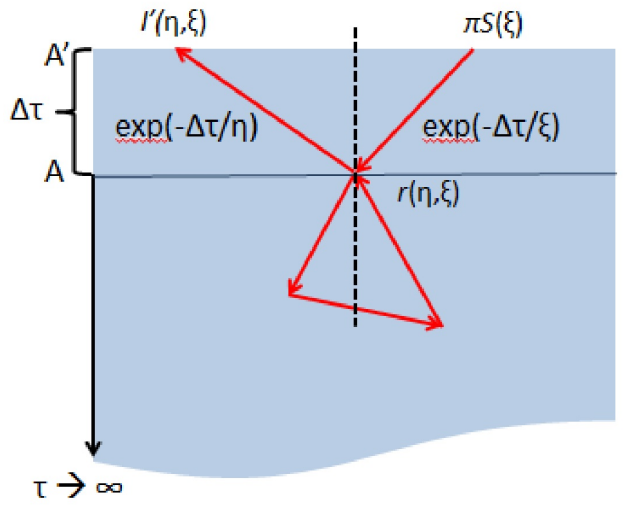
where πS is the plane irradiance measured perpendicular to a collimated beam incident onto the surface of the medium at a polar angle given by $\cos^{-1}(\xi)$, and $I(\eta,\xi)$ is the radiance leaving the medium in direction $\cos^{-1}(\eta)$. [Note: Ambartsumian uses πS for the incident plane irradiance, but defines the radiance reflectance without the π, as seen above.] This geometry is illustrated in Fig. 4(A) below. The surface of the medium is denoted by A. τ is optical depth measured downward from τ = 0 at surface A. The red arrows represent light incident onto the surface of the medium, being scattered and absorbed within the medium, and leaving the medium.



(A) The original problem.

Fig. 4(A). The reflectance problem to be solved.

Ambartsumian then supposes that a thin layer of optical thickness $\Delta\tau$ is added to the original medium, as seen in Fig. 4(B) below. The new surface is at A′.



(B) The medium with a thin layer added.

Fig. 4(B). The original medium with a thin layer of the same materia added.

$\Delta\tau$ is assumed to be thin enough that only terms of first order in $\Delta\tau$ need to be retained in the development to follow. Now what is the radiance leaving the surface at A′? The incident irradiance $\pi S(\xi)$ is unchanged. However, as this irradiance travels through the thin layer, it will be attenuated by a factor of $\exp(-\Delta\tau/\xi)$. The remaining irradiance reaching the depth of the original surface at A will be reflected by the factor $r(\eta,\xi)$ just as before. Then the radiance leaving A will be attenuated by a factor $\exp(-\Delta\tau/\eta)$ on the way back to surface A′. Thus the new radiance $I'(\eta,\xi)$ leaving surface A′ will be

$$\begin{aligned} I'(\eta,\xi) &= \exp(-\Delta\tau/\eta)\, r(\eta,\xi) \exp(-\Delta\tau/\xi)\, S \\ &= r(\eta,\xi)\left(1 - \frac{\Delta\tau}{\eta} + \cdots\right)\left(1 - \frac{\Delta\tau}{\xi} + \cdots\right) S \\ &\approx r(\eta,\xi)\left(1 - \frac{\Delta\tau}{\eta} - \frac{\Delta\tau}{\xi}\right) S, \end{aligned}$$

after dropping terms of order $(\Delta\tau)^2$ and higher. This is the radiance leaving surface A′ due to light that travels through layer $\Delta\tau$ without scattering.

However, the new layer $\Delta\tau$ will also scatter light. Proceeding in a fashion similar to that just seen, Ambartsumian considers 4 cases for scattering with in $\Delta\tau$, which are illustrated in Fig. 5.

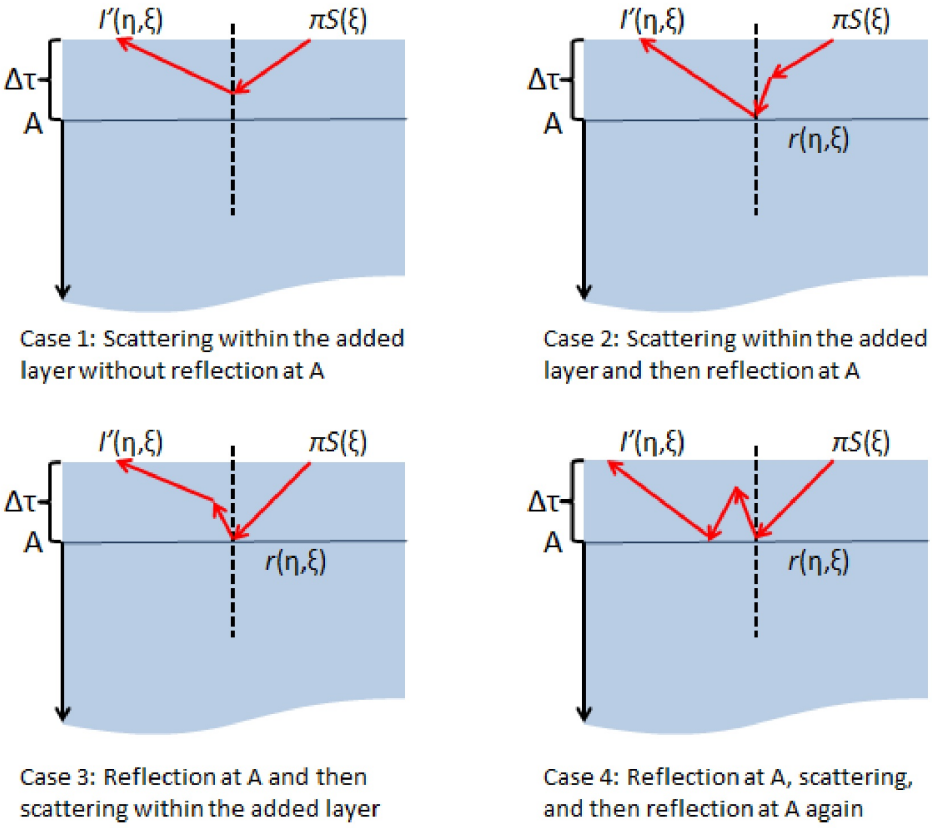


Fig. 5. Illustration of the four additional cases considered by Ambartsumian. All other orders of scattering and reflection involve terms of higher that first order in Δτ.

Case 1. The layer Δτ scatters a part of the direct beam into direction η. This gives a contribution

$$\frac{\lambda}{4}\frac{\Delta\tau}{\eta}S$$

Here λ is the scattering coefficient, and the phase function is assumed to be isotropic.

Case 2. Part of the light is scattered by layer Δτ towards A, where it is partly reflected by A into direction η. This gives a contribution

$$\frac{\lambda}{2}\Delta\tau\, S\int_0^1 r(\eta,\zeta)\frac{d\zeta}{\zeta}\,.$$

Case 3. The layer $\Delta\tau$ scatters the light reflected by A. This gives a contribution

$$\frac{\lambda}{2}\frac{\Delta\tau}{\eta} S \int_0^1 r(\zeta,\xi)\, d\zeta \ .$$

Case 4. Part of the light reflected from A is scattered back by layer $\Delta\tau$ and again is partly reflected from A. This gives a contribution

$$\lambda\,\Delta\tau\, S \int_0^1 r(\zeta,\xi)\, d\zeta \int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'} \ .$$

There are an infinite number of additional possibilities for light to be scattered multiple times within layer $\Delta\tau$, but those all involve contributions of order $(\Delta\tau)^2$ and higher. [Full disclosure: I haven't been able to re-derive the exact same forms as Ambartsumian's terms for Cases 1 to 4. He does not give a figure, so I may be misunderstanding the details of his ray-tracing geometry. Someone help me out!! See also the comment in Chandrasekhar, below.]

Ambartsumian now makes the key observation: *Because the original medium was optically infinitely deep, the medium with the added layer will reflect exactly the same radiance as the original medium.* That is, the total $I'(\eta,\xi)$ from the contributions shown above equals the original $I(\eta,\xi)$. Thus, by Eq. (9) above, the original reflectance is the same as the total reflectance of the five terms due to the added layer:

$$r(\eta,\xi) = r(\eta,\xi)\left(1 - \frac{\Delta\tau}{\eta} - \frac{\Delta\tau}{\xi}\right) + \frac{\lambda}{4}\frac{\Delta\tau}{\eta} + \frac{\lambda}{2}\Delta\tau \int_0^1 r(\eta,\zeta)\,\frac{d\zeta}{\zeta}$$
$$+ \frac{\lambda}{2}\frac{\Delta\tau}{\eta}\int_0^1 r(\zeta,\xi)\, d\zeta + \lambda\,\Delta\tau \int_0^1 r(\zeta,\xi)\, d\zeta \int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'}$$

Note that the S has cancelled out when writing the equation for the reflected radiance. This equation can be rewritten as

$$\left(\frac{1}{\eta} + \frac{1}{\xi}\right) r(\eta,\xi) = \frac{\lambda}{4}\left[\frac{1}{\eta} + 2\int_0^1 r(\eta,\zeta)\,\frac{d\zeta}{\zeta} + \frac{2}{\eta}\int_0^1 r(\zeta,\xi)\, d\zeta + 4\int_0^1 r(\zeta,\xi)\, d\zeta \int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'}\right]$$

Note that the $\Delta\tau$ has now cancelled out.

This is a remarkable equation. This is an integral functional equation for the unknown reflectance $r(\eta,\xi)$, given the IOPs via the scattering coefficient λ and the assumed isotropic phase

function, and the desired incident and final angles. *This equation allows the computation of the radiance reflectance of the optically deep medium without the need to solve the radiative transfer equation within the medium!* This must be compared with the solutions of Lommel and Chwolson: They found integral equations for the radiance *within* the medium, which required integrations of their equations throughout the medium (recall Eq. 6 in the Lommel section and Eq. 8 in the Chwolson section). To compute the radiance leaving the medium, they had to first solve their integral equations *throughout the medium*, and then evaluate that solution at the surface of the medium (at depth $a = 0$ in their equations). They could then take the ratio of the radiance leaving the medium to the incident irradiance, and obtain the reflectance of the medium.

Ambartsumian further simplifies this last equation. Defining

$$R(\eta,\xi) = \frac{4\eta}{\lambda}\, r(\eta,\xi)\,, \qquad \text{(Ambar. Eq. 1)}$$

gives a more symmetric equation for the reflectance:

$$\left(\frac{1}{\eta} + \frac{1}{\xi}\right) R(\eta,\xi) = 1 + \frac{\lambda}{2}\int_0^1 R(\eta,\zeta)\,\frac{d\zeta}{\zeta} + \frac{\lambda}{2}\int_0^1 R(\zeta,\xi)\,\frac{d\zeta}{\zeta} + \frac{\lambda^2}{4}\int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'}\int_0^1 R(\zeta,\xi)\,\frac{d\zeta}{\zeta} \qquad \text{(Ambar. Eq. 2)}$$

He then notes that the resulting equation is symmetric in ξ and η, which further simplifies the integral equation. He then defines

$$\varphi(\eta) = 1 + \frac{\lambda}{2}\int_0^1 R(\eta,\zeta)\,\frac{d\zeta}{\zeta}\,, \qquad \text{(Ambar. Eq. 5)}$$

and shows that

$$R(\eta,\xi) = \frac{\varphi(\eta)\,\varphi(\xi)}{\frac{1}{\eta} + \frac{1}{\xi}}\,. \qquad \text{(Ambar. Eq. 6)}$$

Finally, he arrives at an integral equation for the auxillary function φ:

$$\varphi(\eta) = 1 + \frac{\lambda}{2}\,\eta\,\varphi(\eta)\int_0^1 \frac{\varphi(\xi)\,d\xi}{\eta + \xi}\,. \qquad \text{(Ambar. Eq. 7)}$$

To summarize: To compute the radiance reflectance of an optically deep medium, first solve Ambartsumian's Eq. (7) for φ. Then use the solution φ in his Eq. (6) to get $R(\eta,\xi)$, and finally use $R(\eta,\xi)$ in his Eq. (1) to get the desired radiance reflectance $r(\eta,\xi)$. His Eq. (7) must of course be solved numerically, but that is a much simpler problem than the solution of Eq. (6) of the Lommel section or Eq. (8) of the Chwolson section. Ambartsumian says that "the equation 7 is

easily solved numerically by means of successive approximations,..."

The final page of this paper comments that

> "The method exposed above can be applied not only to media of infinite depth, but also to layers of finite optical thickness τ, enclosed between two parallel planes A and B. In this case, however, we consider not only the function of diffuse reflection $r(\eta,\xi)$ but also a function $s(\eta,\xi)$, which describes the diffuse transparency, i.e., the part of the light which enters via A and leaves the medium via B. To use invariance, we add a thin layer $\Delta\tau$ on the side A and subtract the same layer on the side B."

He then presents a pair of coupled equations for auxillary functions $\varphi(\eta)$ and $\psi(\eta)$, which replace the equation for $\varphi(\eta)$ seem above, and which then give $r(\eta,\xi)$ and $s(\eta,\xi)$.

Ambartsumian's 1942 paper addressed the reflectance problem starting with Milne's equation. His 1944 paper further extended the formalism of the 1943 paper discussed here to the case of anisotropic phase functions.

Ambartsumian called his idea the "principle of invariance" because the reflectance of the homogeneous medium is invariant to the addition of the thin layer. This idea, after decades of further development is now called "invariant imbedding theory." That is, the original problem is imbedded in a new problem, while leaving the reflectance (and/or transmittance) invariant. It was quickly recognized that this seminal idea enabled the relatively easy numerical solution of a wide variety of problems, which had previously been quite difficult to solve in pre-computer days. That is to say, solving Arbartsumian's Eq. (7) is much easier than solving Lommel's or Chwolson's equations. Consequently, Ambartsumian was awarded the Stalin Prize in 1946 for this paper. He went on to win many other honors both within the USSR and internationally, and he circulated in the highest ranks of the Soviet government, always working as a champion for science. In his later years, he said that it was his principle of invariance that was the most gratifying of all his contributions to science.

Введем функцию $R(\eta, \xi)$, определенную через

$$r(\eta, \xi) = \frac{\lambda}{4\eta} R(\eta, \xi). \tag{1}$$

Тогда для $R(\eta, \xi)$ имеем функциональное уравнение

$$\left(\frac{1}{\eta} + \frac{1}{\xi}\right) R(\eta, \xi) = 1 + \frac{\lambda}{2} \int_0^1 R(\eta, \zeta) \frac{d\zeta}{\zeta} +$$

$$+ \frac{\lambda}{2} \int_0^1 R(\zeta, \xi) \frac{d\zeta}{\zeta} + \frac{\lambda^2}{4} \int_0^1 R(\eta, \zeta') \frac{d\zeta'}{\zeta'} \int_0^1 R(\zeta, \xi) \frac{d\zeta}{\zeta}. \tag{2}$$

Очевидно, что если этому уравнению удовлетворяет функция $R(\eta, \xi)$, то ему же должна удовлетворять функция $R(\xi, \eta)$. Так как наша физическая задача должна иметь только одно решение, то возникает мысль искать решение уравнения (2) в виде симметричной функции

$$R(\eta, \xi) = R(\xi, \eta). \tag{3}$$

Но при этом условии правая часть (2) оказывается произведением двух одинаковых функций

$$\left(\frac{1}{\eta} + \frac{1}{\xi}\right) R(\eta, \xi) = \left\{1 + \frac{\lambda}{2} \int_0^1 R(\eta, \zeta) \frac{d\zeta}{\zeta}\right\} \left\{1 + \frac{\lambda}{2} \int_0^1 R(\xi, \zeta) \frac{d\zeta}{\zeta}\right\}. \tag{4}$$

Обозначим

$$\varphi(\eta) = 1 + \frac{\lambda}{2} \int_0^1 R(\eta, \zeta) \frac{d\zeta}{\zeta}. \tag{5}$$

Тогда (4) и (5) сразу дают структуру функций $R(\eta, \xi)$ и $r(\eta, \xi)$:

$$R(\eta, \xi) = \frac{\varphi(\eta)\varphi(\xi)}{\frac{1}{\eta} + \frac{1}{\xi}}; \quad r(\eta, \xi) = \frac{\lambda}{4} \xi \frac{\varphi(\eta)\varphi(\xi)}{\eta + \xi}. \tag{6}$$

Подстановка (6) в (5) дает уравнение для функции $\varphi(\eta)$

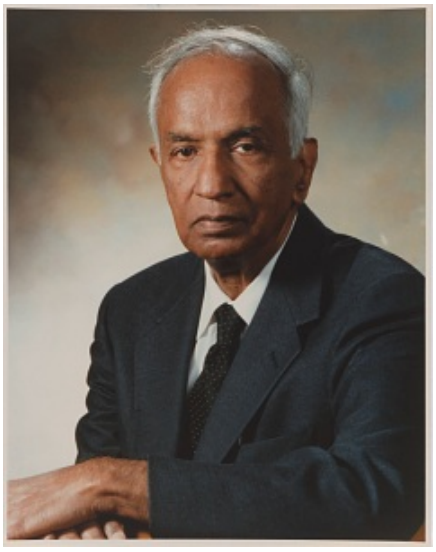
$$\varphi(\eta) = 1 + \frac{\lambda}{2} \eta \varphi(\eta) \int_0^1 \frac{\varphi(\xi)\, d\xi}{\eta + \xi}. \tag{7}$$

Итак, мы приходим к выводу: функция $r(\eta, \xi)$, характеризующая отражательную способность, имеет структуру, выражаемую формулой (6). Функция $\varphi(\eta)$ определяется при этом функциональным уравнением (7).

The page from Ambartsumian (1943) showing the development outlined above. This is surely one of the most important pages in the history of radiative transfer theory.

Sobolev. Viktor Viktorovich Sobolev (Виктор Викторович Соболев) (1915-1999) was a student of Ambartsumian, and he was evacuated to Tartarstan along with his mentor. He had a long and influential career and did much work on radiative transfer applied to astrophysical problems. He once said, "A scientist is not one who is engaged in science, but one who cannot but do it." I could not agree more. (Note that V. V. Sobolev is a different person than S. L. Sobolev, an equally famous mathematician.)

Sobolev was Ambartsumian's Ph.D. student. Sobolev continued Ambartsumian's work and went on to develop techniques for RTT in expanding gases (such as occur in a planetary nebula around stellar nova), inclusion of polarization, problems with anisotropic or inelastic scattering, time-dependent RRT, and RTT in spherical geometries. He developed a number of important solution techniques for his equations. He was the core of what is sometimes called "The Sobolev School of Astrophysics." Although Sobolev is almost a deity in astrophysics, his formulations of RTT are not usually applicable to oceanographic problems, so he is little known to oceanographers. I have included him in these notes because of his contributions to the overall theory of radiative transfer. His contributions to astrophysical RTT are reviewed in Nagirner (2015).



Chandrasekhar. Subrahmanyan Chandrasekhar (1910-1995) was one of India's most famous scientists. He was the nephew of C. V. Raman (who received the Nobel prize in 1930 for his discovery and explanation of what is now called Raman scattering). Chandrasekhar spent most of his career at the University of Chicago, where he worked on astrophysical problems. He received the Nobel Prize in Physics in 1983 for his work on the structure and evolution of stars. He is remembered today for working out (at the age of only 20) the maximum mass of a stable white dwarf star, now known as the Chandrasekhar limit; a more massive star is destined to become a black hole. His treatise *Radiative Transfer*, published in 1950, remains in print today and is a standard text for RTT applied to stellar atmospheres. This book is worth reading, even if just to see how he approached the subject.

One of the topics developed in detail in *Radiative Transfer* is various principles of invariance. Chandrasekhar takes a more general approach, beginning with the radiative transfer equation, than did Ambartsumian, who used heuristic arguments to derive the integral equation seen previously for the reflectance $R(\eta,\xi)$ (which is denoted $S(\mu,\mu_o)$ in Chandrasekhar; see Chapter IV of *Radiative Transfer*). Chandrasekhar's general equation for the reflectance *(Radiative Transfer* §30, Eq. 28) is valid for any phase function. When evaluated for isotropic scattering (his §33, Eq. 38) , Chandrasekhar's integral equation for $S(\mu,\mu_o)$ reduces to Ambartsumian's Eq. (2) for $R(\eta,\xi)$ (except for a minor difference in the treatment of the albedo of single scattering). Chandrasekhar then obtains equations (his §33, Eqs. 40 and 42) that correspond exactly to Ambartsumian's Eqs. (5) and (7) seen above, with the exception that Chandrasekhar uses $H(\mu)$ to denote Ambartsumian's $\varphi(\eta)$. Chandrasekhar then devotes an entire chapter to the properties and evaluation of the $H(\mu)$ functions, and he presents tables of numerical values of $H(\mu)$ as a function of μ and the albedo of single scattering.

Chandrasekhar goes on to give a detailed development of the general (for any phase function) invariance principles for both reflectance and transmittance when layers are added or subtracted, which again reduce to Ambartsumian's equations for the case of isotropic scattering.

Chandrasekhar's work did much to bring Ambartsumian's ideas to the attention of the broad community of radiative transfer theorists, and to make the needed functions ($H(\mu)$, etc) readily available for computations. However, although many of his equations are valid for any phase function, and often include polarization, his solution techniques never extend beyond a Rayleigh

phase function. This is adequate for studies of gaseous stellar atmospheres, which do not contain phytoplankton, but are of little applicability to oceanic problems for which phase functions are highly peaked in the forward-scattering directions. I well remember spending $7.95 for a copy of *Radiative Transfer* when I was a post doc, believing that all knowledge of RTT would be revealed to me by a Nobel Prize laureate. I soon discovered that nothing in the book was of any use in computing underwater radiance distributions in real oceans.

Preisendorfer. Rudolph William Preisendorfer (1927-1986) was an applied mathematician of the first magnitude. He worked in a number of areas including tsunami forecasting and statistical methods for climate prediction, but he is best known for his work in radiative transfer theory applied to optical oceanography, which culminated in his six-volume treatise *Hydrologic Optics* (1977). I worked with him from 1978 to 1986 on the development of numerical methods for solving the radiative transfer equation using invariant imbedding. The result of that collaboration is the well-known HydroLight software, which is widely used to this day.

Photo from Curtis Mobley's files.

Preisendorfer's first love was "hydrologic optics," as he called the general subject of radiative transfer in the aquatic environment. After working as a summer student with S. Q. Duntley on underwater optics problems at Lake Winnepesaukee, New Hampshire, he was hooked. His undergratuate thesis at MIT (under Duntley's supervision) was on the analytic solution of a simplified radiative transfer equation, and his Ph.D. dissertation (Preisendorfer, 1957) was titled "A Mathematical Foundation for Radiative Transfer Theory." There followed a series of mathematical papers developing invariant imbedding theory in a form suitable of application to oceanic problems. He once told me of his difficulties in getting these papers published. He would send a paper to an optics journal, only to have it returned without review along with the comment "this is oceanography; you need to submit this to an oceanography journal." He would then resubmit the paper to an oceanography journal, only to be told, "this is mathematics; you need to submit this to a math journal." The math journal would then say, "this is optics, you need to submit this to an optics journal." After several rounds of such rejections, he concluded that the entire refereed journal publication business was run by idiots. He was finally able to get a number of his papers published in the Proceedings of the National Academy of Sciences; these papers all show "Communicated by S. Chandrasekhar." In his later years, Preisendorfer tended to avoid publication in the refereed literature. Instead, he maintained a list of colleagues interested in his work. Anyone could request to be on this list, and if you weren't on the list that was your fault. He would send his latest works to those people, often as photocopies of handwritten notes. I have been told by one of those colleagues that his unpublished work was sometimes incorporated into other people's papers, often without adequate acknowledgment of his contribution. Knowing Rudy, I don't think this bothered him a bit.

Preisendorfer set himself the goal of constructing "an analytical bridge between the mainland of physics and the island of radiative transfer theory," as he worded it on page 389 of his 1965 treatise *Radiative Transfer on Discrete Spaces* (Preisendorfer, 1965). At that time, RTT was still a *phenomenological* theory, meaning that the governing radiative transfer equations were based on heuristic arguments and physical intuition, rather than being derived from more fundamental physics such as Maxwell's equations for the electric and magnetic fields that give the physical description of light propagation. Search all you want in physics texts such as Jackson's *Classical Electrodynamics* (1962) and you will find no mention of radiance, nor even of solid angle.

Preisendorfer (and other mathematicians, Richard Bellman in particular) further developed Ambartsumian's fundamental idea to the point that the mathematical connection to Ambartsumian's 1943 paper is almost unrecognizable to the untrained eye. The resulting equations enable the computation of the reflectance and transmittance of inhomogeneous layers of arbitrary thickness and optical properties. This is what is needed for oceanographic applications.

Preisendorfer's equations are the core of the HydroLight radiative transfer software. If you run HydroLight, one of the user inputs is a list of depths where the outputs (the radiance distribution and all other radiometric quantities such as irradiances and reflectances) are to be saved. Those depths define finitely thick layers within the water body, but these layers can have any depth dependence of the absorption and scattering properties, and the particles within the water can have any phase function. If you look deep inside the HydroLight code, it is not solving the integro-differential RTE for the radiance *per se*; it is solving a set of nonlinear ordinary differential equations (Riccati-type equations) for the reflectances and transmittances (in Fourier amplitude form) of the layers defined by the user-requested output depths. These reflectances and transmittances, which include all orders of multiple scattering, once computed are then combined with boundary conditions at the sea surface and bottom to obtain the radiances at the user-requested depths. The development of these equations for reflectances and transmittances, starting with the scalar integro-differential RTE for the radiance, is seen in Chapter 8 *of Light and Water* (Mobley, 1994). The corresponding development for the vector RTE is given in Mobley (2018).

Preisendorfer highlighted the difference in *local* and *global* formulations of physical theories. A local formulation of a physical theory means that the governing equations use spatial derivatives of the quantities of interest at a particular point. The integro-differential RTE is an example of a local formulation of RTT because it involves depth derivatives of the radiance at each depth within the medium. (Another example of a local formulation of a physical law is Maxwell's equations in differential equation form.) A global formulation does not use spatial

derivatives. Ambartsumian's reflectance equation, and Preisendorfer's extensions thereof, compute the reflectance (and transmittance) of entire, finitely or infinitely thick layers "all at once." Thus Schuster's two-flow differential equations are a local formulation of a radiative transfer problem; Ambartsumian's integral equation for the reflectance of a semi-infinite medium is a global formulation.

Preisendorfer's development of the mathematics of RTT culminated in his six-volume, 1,757 page *opus magnum*, *Hydrologic Optics* (Preisendorfer, 1977). In spite of the great care he used in formulating the mathematical and physical foundations of RTT, he never quite succeeded in building the bridge between physics and RTT. That is the subject of the final section of these notes. Nevertheless, Preisendorfer was one of the greatest theoreticians of RTT for two reasons. First, he continued Gershun's attempt to put RTT on a firm mathematical and physical foundation, and he made great progress towards that goal. Second, he brought Ambartsumian's idea of invariance and a global formulation of RTT to its fullest form, valid for almost any optical medium. Without his decades of work to perfect invariant imbedding theory, there would be no HydroLight.

Forgotten and Remembered

Both Lommel's 1887/1889 paper and Chwolson's 1889 paper were physical and mathematical tours de force, which broke much new ground. Their approaches were thoroughly modern. Unfortunately, their papers seem to have attracted little attention and were apparently quickly forgotten. Subsequent authors scarcely ever referenced either paper.

Schuster (1905) makes no reference to either Lommel or Chwolson, so it appears that their papers were either already forgotten or not recognized for their importance. Schuster's 1905 paper is often cited by later authors as being the foundational paper of RTT. For example, in his 1930 *Handbook der Astrophysik* article, Milne did not reference either Lommel or Chwolson but stated that Schuster's 1905 paper was "The pioneering paper on the transfer of radiation by scattering." Duntley (1942) states that "The first fundamental approach to this type of problem [the problem of light propagation in "diffusing materials"] was made by the astronomer Arthur Schuster. In 1905 he published a theory describing the escape of radiation from the self-luminous, foggy atmosphere of a star." Chandrasekhar (1960) says in the preface to his monumental treatise that (after referring to Rayleigh 1871), "However, the subject was given a fresh start under more tractable conditions with Arthur Schuster formulated in 1905 a problem in Radiative Transfer in an attempt to explain the appearance of absorption and emission lines in stellar spectra, and Karl Schwarzschild introduced in 1906 the concept of radiative equilibrium in stellar atmospheres." None of these later papers reference either Lommel or Chwolson.

The only citations to Lommel and Chwolson from the early 1900s that I have found are in a mathematical treatise on the logarithmic integral (Nielsen, 1906), which merely lists their papers in a lengthy bibliography as having used the logarithmic integral for "physical applications," and King (1913a) who cites Lommel as mentioned previously. I have not found any references to Lommel or Chwolson in the classic works of Chandrasehkar, Jerlov, Preisendorfer, Tyler, Duntley, or several other luminaries of their era.

The earliest "modern" reference to Lommel and Chwolson I've found is in van de Hulst (1980), where he says (volume 1, page 127), "It may be noted, however, that the basic integral equation (for which we use the now common name 'Milne equation') ... was already explicitly derived by Lommel (1889) and Chwolson (1890)." I myself had never heard of Lommel until I found a nearly illegible photocopy of his 1889 paper in Preisendorfer's files after his death. I then briefly summarized Lommel's work in *Light and Water* (1994, pages 312-313). At that time, I was unaware of Chwolson; I believe I first saw a reference to Chwolson in one of Mishchenko's books.

In recent years Lommel and Chwolson have finally received the recognition they are due. Nowadays, it seems obligatory to add at least a sentence or two acknowledging the seminal contributions of Lommel and Chwolson to the foundations of RTT. For example, a recent paper by Lessig et al. (2012) on the mathematical formulation of radiance states, when reviewing the history of radiomentric concepts, "Although many prominent scientists and distinguished mathematicians, such as von Helmholtz, Kirchhoff, and Clausius, employed radiometry, the physical and mathematical foundations of the theory remained entrenched in the 18th century. A notable exception is the work by Lommel. He considered radiance as emanating from a volume, an idea that was forgotten shortly afterwards,..." And then a bit later, Lessig et al. say, "Similar results that accounted for scattering were obtained by Chwolson and somewhat later by Schuster, while Schwarzschild studied radiation equilibrium, a question again considered before by Lommel. Based on Schuster's work, Jackson and King developed the transport equations that are in use to this date."

Lommel and Chwolson performed essentially the same physical and mathematical analyses and obtained essentially the same integral forms of the RTE. Their equations are similar to what is seen in *Light and Water* Eq. (5.30), where an integral form of the RTE is obtained by integration the integro-differential Eq. (5.24). They published their papers almost simultaneously, and they were likely unaware of each other. Given the seminal nature of their papers and their nearly simultaneous publications, I personally believe that Lommel and Chwolson should be recognized as "The Fathers of Radiative Transfer Theory."

Completing the Bridge

As has been mentioned, both Gershun and Preisendorfer recognized the need to make RTT a rigorous subdiscipline of physics. This is not something that worries most oceanographers because the equations of phenomenological RTT give answers of sufficient accuracy for the practical problems of optical oceanography and ocean color remote sensing. However, The Deep Thinkers do lie awake at night worrying about esoteric matters physics and mathematics.

Indeed, the introduction of the 1939 English translation of Gershun's *The Light Field* begins with

> Theoretical photometry constitutes a case of "arrested development", and has remained basically unchanged since 1760 while the rest of physics has swept triumphantly ahead. In recent years, however, the increasing needs of modern lighting technique have made the absurdly antiquated concepts of traditional photometric theory more and more untenable.

Ideally there should be a path leading from the most fundamental physical laws governing electromagnetism and light clear through to the simplest of radiative transfer equations such as the scalar RTE solved by HydroLight or the two-flow irradiance equations of Schuster (1905). That path is conceptualized in the figure below.

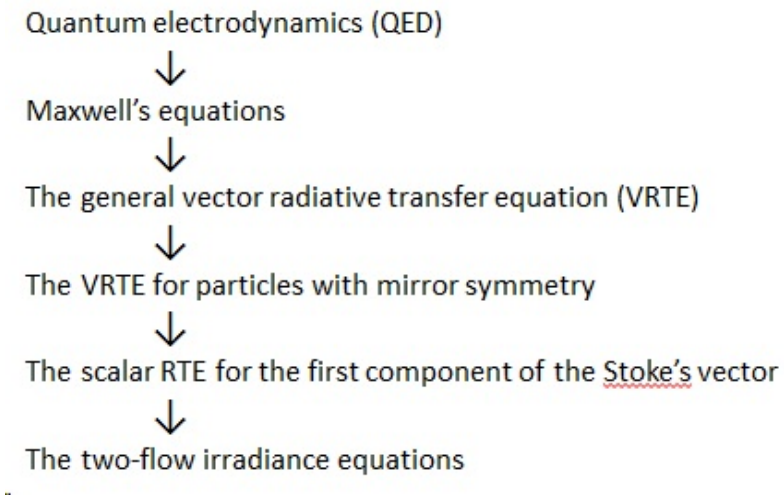


Fig. 6. Steps to a rigorous formulation of RTT.

Quantum electrodynamics (QED) is the fundamental physical theory that explains with (as far as we know) total accuracy the interactions of light and matter, and of electrically charged particles. It was developed in the 1940s and 1950s by Richard Feynman, Julian Schwinger, and Shinichiro Tomonaga, who shared the 1965 Nobel Prize in Physics for their work. Although the

conceptual rules of QED are easy to state, the associated mathematics is outrageously complex, and consequently the theory can be applied only at the level of interactions between elementary particles.

QED is a quantum field theory that describes light and electromagnetism at the level of individual photons, which in QED are viewed as the quantized vibrational modes of the electromagnetic field. It is possible to take a "classical physics limit" of QED to get a classical field theory, in which the electromagnetic field is not quantized. The result is Maxwell's equations, which describe electric and magnetic fields as continuous functions of space and time. The step from QED to Maxwell's Equations was made by Feynman, who, as the story goes, showed that Maxwell's Equations can be derived from QED but never published the derivation.

The "general VRTE" shown in the figure above (and seen below) is a vector radiative transfer equation valid for any medium or assemblage of particles of any size, shape, or orientation. Getting from the general VRTE to a simpler VRTE as commonly used in atmospheric or oceanic optics can be done by any good physicist. Getting from there to a scalar equation is easy, and getting from the scalar RTE to two-flow equations is almost trivial (e.g., *Light and Water* §5.11). The crux of building the bridge from the mainland of physics to the island of RTT is getting from Maxwell's Equations to the general VRTE. That span of the bridge was not completed until the last two decades, and the chief engineer was M. I. Mishchenko.

Mishchenko. Michael I. Mishchenko is a physicist specializing in electromagnetic scattering theory, especially as applied to atmospheric optics. For more than 20 years he has been Senior Scientist at the NASA Goddard Institute of Space Studies. He has authored approximately 300 publications, several highly technical books, and numerous book chapters; these have been cited almost 30,000 times. His contributions to RTT are unfortunately under-appreciated in the optical oceanography community, no doubt because the price of admission for reading his papers is a very deep understanding of electromagnetic scattering theory and the associate mathematics. His resume at https://www.giss.nasa.gov/staff/mmishchenko/mishchenko.html will leave you in awe. In my humble opinion, he holds the title of Deepest Thinker in Radiative Transfer Theory.

If ever there is a place in science where ignorance is bliss, it is in RTT. Mishchenko has published a series of papers on the foundations of RTT, which show that phenomenological RTT is "a tissue of fallacies," to paraphrase Bertrand Russell's description of Newton's calculus. A good place to start understanding the deficiencies of phenomenological RTT and Mishchenko's contributions to RTT is his two recent reviews "125 Years of Radiative Transfer: Enduring Triumphs and Persisting Misconceptions" (Mishchenko, 2013) and "Directional radiometry and radiative transfer: The convoluted path from centuries-old phenomenology to physical optics" (Mishchenko, 2014). Even if you can't follow the math (and I myself seldom can), you need to read these two papers to get an ideal of the fundamental problems with phenomenological RTT, and of the resolution of those problems.

The fundamental problem with RTT as historically formulated is that the concept of radiance (specific intensity) itself is inconsistent with electromagnetic theory as described by Maxwell's equations. The radiance is usually described as giving the "angular distribution of radiant (electromagnetic, light) energy flow," or something to that effect. The usual derivation of the scalar RTE (as seen in *Light and Water* Chapter5, for example) is then based on the idea that the radiance describes the radiant energy traveling through each point in space in all directions. The RTE is then derived by invoking conservation of energy via various processes (absorption, scattering, emission) that affect the radiance. The same arguments can be applied at the vector level to include the effects of polarization. However, this approach is incorrect because it is not radiant *energy* that interacts with matter, it is *electric and magnetic fields*.

Figure 7 below shows the inconsistency of the energy vs electromagnetic field view points of radiometry. The red lines represent two plane waves of light traveling *in vacuo* in directions $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ approaching a well collimated radiometer (WCR) whose field of view is shown by the dotted lines; the WRC has a solid angle of $\Delta\Omega$ and a collector of area ΔA. *Neither of these incident plane waves will be detected by the radiometer because their directions of propagation are outside the field of view of the radiometer.* According to Maxwell's equations, these plane waves are described by their electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$, which as drawn are pointing out of the paper towards the reader, and by their magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$, which are perpendicular to the electric fields. These field vectors are shown in blue. The energy in these plane waves is given (in SI units) by their respective Poynting vectors, $\mathbf{S}_1 = (1/\mu_o)\,\mathbf{E}_1 \times \mathbf{B}_1$ and $\mathbf{S}_2 = (1/\mu_o)\,\mathbf{E}_2 \times \mathbf{B}_2$, where μ_o is the permeability of free space and $\times$ is the vector cross product. These Poynting vectors are shown in green; these vectors point in the same directions as the propagation directions $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$. At the entrance of the radiometer, these fields combine by vector addition to give the total electric and magnetic fields, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$. The corresponding total Poynting vector $\mathbf{S} = (1/\mu_o)\,\mathbf{E} \times \mathbf{B}$ points directly into the radiometer. *Thus according to electromagnetic theory, the net flow of energy is directly into the radiometer, but the incident light is not detected.* We are forced to conclude that a WCR does not actually detect the net flow of radiant energy as described by Maxwell's equations! This is contradictory to the viewpoint of phenomenological RTE, which is based on energy arguments and views a radiometer as an instrument that collects radiant energy traveling into the field of view of the instrument.

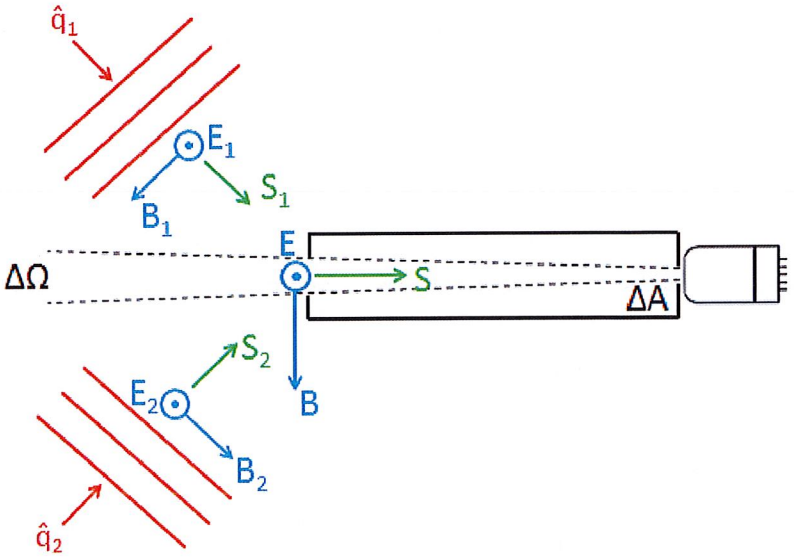


Fig. 7. Illustration of the inconsistency of phenomenological RTT and electromagnetic theory. Based on Fig.12d of Mishchenko (2014).

The preceding argument shows that something is fundamentally wrong with the heuristic arguments of energy-based, phenomenological RTT. My presentation above is intended to pique your interest; I refer you to Mishchenko's papers for a quantitative analysis and resolution of these problems.

As Mishchenko emphasizes, in practice we need to solve two well posed physical problems:

(1) We need to compute the radiant energy budget of a volume of material (say a volume of ocean water). This will allow us, for example, to compute the heating of the water, and to compute how much light energy is available for photosynthesis.

(2) We need to understand exactly what a "well collimated radiometer" measures (e.g., a common "Gershun tube" type of radiometer). This will allow us to properly interpret our measurements and to understand the relation between what is in the water and the optical quantities being measured, which is the basis of remote sensing.

Mishchenko shows that both of these problems can be solved without resorting to heuristic arguments or even introducing the concept of radiance.

Consider a volume containing a large number of scattering and absorbing particles whose single-particle absorption and scattering properties are known. Then, invoking only the fundamental laws of electromagnetism, and after a great deal of outrageously abstract math, Mishchenko arrives at the equation (Mishchenko, 2013; cf. 2002, 2014)

$$\hat{\mathbf{q}}\cdot\nabla\tilde{\mathbf{I}}(\mathbf{r},\hat{\mathbf{q}}) = -n_{\mathrm{o}}\langle\mathbf{K}(\hat{\mathbf{q}},\xi)\rangle_{\xi}\,\tilde{\mathbf{I}}(\mathbf{r},\hat{\mathbf{q}})+n_{\mathrm{o}}\int_{4\pi} d\hat{\mathbf{q}}'\langle\mathbf{Z}(\hat{\mathbf{q}},\hat{\mathbf{q}}',\xi)\rangle_{\xi}\,\tilde{\mathbf{I}}(\mathbf{r},\hat{\mathbf{q}}') \quad \text{(Mish 2013, Eq.4)}$$

Here $\hat{\mathbf{q}}$ is a unit direction vector, $\mathbf{r}$ is the location in 3D space, n_0 is the average number of scattering particles per unit volume, $\mathbf{K}$ is the 4x4 real-valued, single-particle extinction matrix, ξ labels the microphysical state of a particle (orientation, etc), $\langle\rangle_{\xi}$ indicates the average over all microphysical states, and $\mathbf{Z}$ is the 4x4 real-valued, single-particle phase matrix, which is also averaged over all microphysical states. In general, for irregularly shaped particles without any symmetry distributed in any random or non-random orientations and having any optical properties such as dichroism, all 16 elements of $\mathbf{K}$ and $\mathbf{Z}$ will be non-zero. $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q})$ is a 4 x 1 column vector.

In addition, Mishchenko shows that the quantity measured by a WCR is

$$\Delta A \, \Delta\Omega \, \tilde{\mathbf{I}}(\mathbf{r},\hat{\mathbf{q}}) \qquad \text{(Mish 2013, Eq. 3)}$$

and that the time-averaged Poynting vector is given by

$$\langle \mathbf{S}(\mathbf{r},t) \rangle = \int_{4\pi} d\hat{\mathbf{q}} \, \hat{\mathbf{q}} \, \tilde{I}(\mathbf{r},\hat{\mathbf{q}}) \qquad \text{(Mish 2013, Eq. 5)}$$

where $\tilde{I}(\mathbf{r},\mathbf{q})$ is the first element of the 4 element vector $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q})$, $\mathbf{S}(\mathbf{r},t)$ is the Poynting vector at instantaneous time t, and $\langle \mathbf{S}(\mathbf{r},t) \rangle$ denotes a time average over a sufficiently long time.

Mishchenko's Eq. 4 seen above is the general VRTE referred to in Fig. 6 above. This equation looks just like the vector RTE de`rived by heuristic arguments. However, this equation was derived directly from electromagnetic theory without any use of heuristic arguments or any mention of radiance, and the interpretation of this equation is totally different. I can do no better than to quote Mishchenko (2013, page 16):

> One might claim that Eq. (4) is the standard RTE postulated in the phenomenological RTT based on vague energy-conservation and directional-energy-propagation arguments. Furthermore, one might equate $\tilde{\mathbf{I}}(\mathbf{r},\hat{\mathbf{q}})$ with the phenomenological radiance and thereby attribute to it primordial physical significance as the quantity specifying the angular distribution of electromagnetic energy flow at the point $\mathbf{r}$ over all propagation directions $\hat{\mathbf{q}} \in 4\pi$.
>
> The microphysical approach to radiative transfer shows that this interpretation of Eq. (4) and the quantity $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q})$ is thoroughly incorrect. The quantity $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q})$ does enter the formula for the time-averaged Poynting vector. However, we have already seen that even the Poynting vector cannot be legitimately claimed to specify the direction of the time-averaged electromagnetic energy flow, and so there is even less justification for ascribing any "directional energy" content to the specific intensity. The quantity $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q})$ is nothing but a formal solution of the intermediate equation (4) and appears as a byproduct of the purely mathematical derivation of Eqs. (3) and (5) from the Maxwell equations.

In other words, we can solve the physical problems (1) and (2) above as follows. We first solve Eq. (4) above for the vector quantity $\tilde{\mathbf{I}}(\mathbf{r},\hat{\mathbf{q}})$, which indeed does have the same units as the phenomenological radiance. We can then compute the energy balance of problem (1) above via Eq. (5) and Poynting's theorem, which says that the net energy entering or leaving a volume of space is the integral of the component of $\langle \mathbf{S}(\mathbf{r},t) \rangle$ normal to the surface enclosing the region,

integrated over the surface enclosing the volume. The answer to Problem (2) above is then given by Eq. (3), where the $\hat{\mathbf{q}}$ in that equation is the set of directions within the field of view of the instrument.

Mishchenko's Eq. (4) above is the long-awaited link between Maxwell's equations and RTT. Construction of this link required neither the heuristic arguments of phenomenological RRT nor any mention of radiance as historically defined.

Simplifications of the General VRTE

[This section is condensed from the Ocean Optics Web Book page at www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/the_vrte_for_mirrorsymmetric_media .]

This section goes beyond the needs of a historical review, but nevertheless is justified in order to finish the story of how electromagnetic theory is connected to the radiative transfer equations commonly used by optical oceanographers.

The general VRTE seen in Mishchenko's Eq. (4) above is applicable to almost any medium containing particles that are not so densely packed that the "far-field" approximation fails. That is to say, the particles are far enough away from each other that scattering from each particle is independent of scattering by its neighbors. This is the case for phytoplankton and other types of particles in the ocean, which on average are separated by a hundred or more wavelengths of visible light. Moreover, oceanic particles often have a degree of symmetry, are seldom dichroic, and the overall medium does not transmit light differently in different directions (as can a crystal). Thus the generality of his Eq. (4) is seldom needed in applications to oceanography problems.

The general VRTE simplifies greatly if the scattering particles are randomly oriented and have mirror symmetry. A particle has mirror symmetry if a translation and/or rotation of the mirror-reflected particle can make it congruent with the original particle. This is illustrated in the figure below.

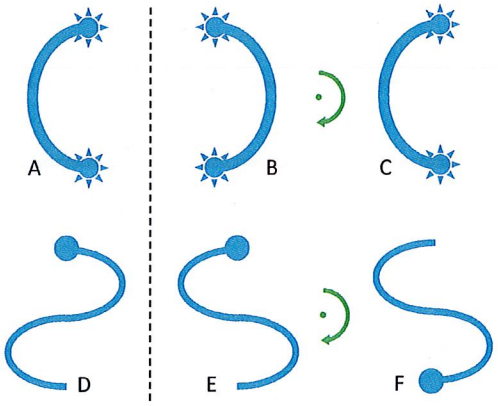


Illustration of mirror symmetry. Panels A and D represent two particles. The dashed line is a mirror. The second column is the mirror image of the first column. The last column is the mirror image rotated by 180 deg about an axis normal to the figure, as illustrated by the green arrows. Top row: a mirror-symmetric particle; bottom row: a particle that is not mirror-symmetric.

If the scattering medium consists of randomly oriented, mirror symmetric particles, then the bulk medium is directionally isotropic and mirror symmetric. In this case, the 4 x4 extinction matrix becomes diagonal, with each element equal to K_{11}. There is then a common extinction coefficient for all states of polarization and directions of propagation:

$$\mathbf{K}(\hat{\mathbf{q}}) = K_{11}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The quantity $n_0 K_{11}$ is the oceanographer's beam attenuation coefficient c.

By definition the phase matrix $\mathbf{Z}(\hat{\mathbf{q}},\hat{\mathbf{q}}')$ scatters light from the incident meridian plane (defined by the incident direction $\hat{\mathbf{q}}'$ in a convenient coordinate system) to the final meridian plane (defined by the scattered direction $\hat{\mathbf{q}}$). It is convenient to write $\mathbf{Z}(\hat{\mathbf{q}},\hat{\mathbf{q}}')$ as the product of three matrices that

1. Transform the initial (unscattered) vector $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q}')$ from the incident meridian plane to the scattering plane, which is the plane containing the incident $\hat{\mathbf{q}}'$ and scattered $\hat{\mathbf{q}}$ directions,

2. Scatter the vector from incident direction $\hat{\mathbf{q}}'$ to scattered direction $\hat{\mathbf{q}}$, with calculations performed in the scattering plane, and

3. Transform the final vector $\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q})$ from the scattering plane to the final meridian plane.

When this is done, the phase matrix is written as

$$\mathbf{Z}(\hat{\mathbf{q}},\hat{\mathbf{q}}') = \mathbf{R}(\alpha)\,\mathbf{M}(\hat{\mathbf{q}},\hat{\mathbf{q}})\,\mathbf{R}(\alpha')$$

Here $\mathbf{R}(\alpha')$ is a 4x4 matrix that transforms ("rotates" through an angle α') the incident vector into the scattering plane, $\mathbf{M}(\hat{\mathbf{q}},\hat{\mathbf{q}})$ is a 4 x 4matrix, called the *scattering matrix*, which by definition scatters the incident vector to the final vector, with both expressed in the scattering plane; and $\mathbf{R}(\alpha)$ is a 4 x4 matrix that rotates the final (scattered) vector from the scattering plane to the final meridian plane.

For an isotropic, mirror-symmetric medium, the scattering matrix becomes block symmetric with only six independent elements. Moreover, the scattering then depends only on the included angle $\psi = \cos^{-1}(\hat{\mathbf{q}}'\cdot\hat{\mathbf{q}})$ (the scattering angle) between directions $\hat{\mathbf{q}}'$ and $\hat{\mathbf{q}}$, and not on $\hat{\mathbf{q}}'$ and $\hat{\mathbf{q}}$ individually . (For the form of this $\mathbf{M}(\psi)$ see Mishchenko et al. (2002) or the Ocean Optics Web

Book page cited above.)

For many applications it is acceptable to model the ocean as a plane parallel medium for which quantities depend spatially only on the depth z. The left side of the general VRTE seen in Eq. (4) above then becomes

$$\hat{\mathbf{q}}\cdot\nabla\tilde{\mathbf{I}}(\mathbf{r},\mathbf{q}) = \cos\theta\frac{d}{dz}\tilde{\mathbf{I}}(z,\theta,\phi)$$

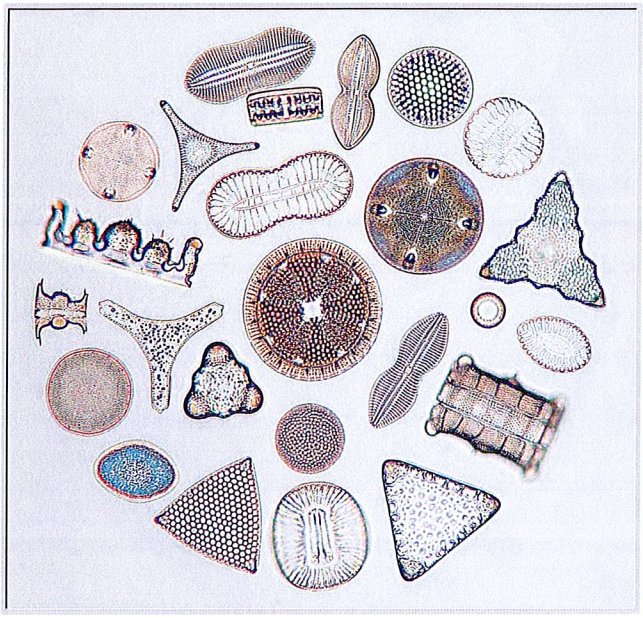
where polar and azimuthal directions θ and ϕ are now used to specific the direction of the unit vector $\hat{\mathbf{q}}$.

These simplifications lead to the 1D VRTE that is commonly used in atmospheric and oceanic optics:

$$\cos\theta\frac{d}{dz}\tilde{\mathbf{I}}(z,\theta,\phi) = -c(z)\,\tilde{\mathbf{I}}(z,\theta,\phi) + \iint_{4\pi}\mathbf{R}(\alpha)\,\mathbf{M}(z,\psi)\,\mathbf{R}(\alpha')\,\tilde{\mathbf{I}}(z,\theta',\phi')\,d\Omega(\theta',\phi')$$

This is as far as we should go in simplfying the RTE if polarization is to be included. This equation again has the same form as the 1D VRTE derived by heuristic arguments, but now the derivation comes via an unbroken chain of physics reaching back to Maxwell's equations.

An obvious question is whether or not the assumption of mirror symmetric particles is justified for oceanographic applications. The figure below shows a collection of oceanic diatoms (arranged for artistic purposes). Many of these are clearly not spherical (contrary to what is often assumed by modelers who can't wean themselves away from Mie theory), but they all appear to be mirror-symmetric to a good approximation. The same holds true for many other species of phytoplankton which, if not roughly spherical, at least have bilateral symmetry. Likewise, atmospheric particles such as fog droplets, snowflakes, and ice crystals are often mirror symmetric.



Diatom shapes. Photo Kreispraeparat-25-G, from http://www.mikroskopie-ph.de.

A final step in simplifying the VRTE can be made by extracting an equation for the first component of $\tilde{\mathbf{I}}(z,\theta,\phi)$, which historically has been called the radiance $L(z,\theta,\phi)$. This is best done by considering the actual form of the block-diagonal scattering matrix $\mathbf{M}(\psi)$, incorporating the explicit forms of the rotation matrices, and writing out the resulting equation for $L(z,\theta,\phi)$. This is very instructive because it allows for an estimate to be made of the error that results if polarization is ignored. This takes us beyond the purview of notes on the history of RTT, but the development can be seen at
http://www.oceanopticsbook.info/view/radiative_transfer_theory/level_2/the_scalar_radiative_transfer_equation

Closing Thoughts

The ancients were of course aware of the *effects* of radiative transfer through absorbing and scattering media. Galileo, for example, observed that "...distant mountains appear blue and somewhat paler, though in reality they are just as dark as those that are nearer to us,..." [Quoted in Reeves (1997), page 116]

However, as we have seen, quantitative photometry arose from visual observations of screens illuminated by a candle in the work of Bouguer, with the corresponding mathematical formulation tracing back to Lambert.

The pioneering papers of Lommel and Chwolson were unfortunately soon forgotten, and it was several decades before physics and mathematics of equal sophistication again appeared in the work of King. Wheels often get reinvented. King gets credit for developing the first RTE for an arbitrary phase function.

Meanwhile, Schuster and Schwarzshild used simple equations to obtain approximate solutions to a variety of astrophysical problems. Their work was especially remarkable in that they used simple optical observations of solar spectra and limb darkening to deduce basic features of the temperature profile within the solar atmosphere and to understand the role of energy transfer by radiation in the Sun's outer layers. Their work constitutes an elegant solution of a complicated radiative transfer inverse problem when reduced to its simplest analytical form.

Schuster's 1905 paper explained bright and dark spectral lines by considering scattering and a blackbody source term in the two-flow equations. He showed that he could obtain either dark or bright spectral lines depending on the relative contributions of absorption and scattering and on the temperature profile. His contribution to RTT was the introduction of the two-flow equations for plane irradiance.

Schwarzschild 1906 considered absorption and a blackbody source term in the two-flow equations (ignoring scattering) and in a radiance-level RTE to deduce the thermal structure of the sun's atmosphere. His contribution to RTT was the introduction of a differential form of the RTE for radiance.

Milne revisited the limb-darkening problem with a more detailed analysis than that of Schwarzschild. Rather than solving for the radiance leaving the solar atmosphere using assumed temperature profiles (as did Schwarzschild), Milne derived an equation for the depth dependence of the blackbody source term (i.e., for the depth dependence of the temperature) corresponding to

the requirement than the Sun's atmosphere be in radiative equilibrium.

Gans then considered polarization in the first paper to treat RTT in the ocean. His paper was also remarkable in that it dealt with electric fields, rather than with energy.

Ambartsumian re-examined the problem of the diffuse reflectance of an optically deep medium, which had been addressed by both Lommel and Chwolson over four decades before. However, Ambartsumian found an entirely new approach to the problem, which would now be called a global formulation of RTT. Fortunately Ambartzumian did not suffer the same fate of obscurity as did Lommel and Chwolson.

Ambartsumian's seminal invariance ideas received a wide audience via Chandrasekhar's treatise of 1950. Preisendorfer then extended the invariance concepts to give them great generality so that they could be applied to inhomogeneous media with arbitrary phase functions—just what is required for oceanographic applications. Preisendorfer's equations are the mathematical core of the HydroLight radiative transfer code, which for over two decades has found wide use in optical oceanography and ocean-color remote sensing.

Gershun made the first attempt to put RTT on a firm physical foundation. Preisendorfer again recognized the need to connect phenomenological RTT with fundamental physics. Although he was especially rigorous in his mathematics and made great progress, the connection was not fully completed until the exceptionally sophisticated work of Mischenko and his colleagues in the twenty-first century. It thus took 70 years after Gershun and almost half a century after the first of Preisendorfer's attempts before RTT was fully incorporated into physics.

Although the contradictions and inconsistencies of phenomenological RTT have now been exposed and resolved by Mishchenko, the old conceptualizations, terminology, and misunderstandings are not going to disappear any time soon. It is simply too convenient to think in terms of energy-based radiance as formulated since the days of Planck. Mishchenko recognizes the uphill battle to be fought. In the closing section of Mishchenko (2014), he quotes Born and Wolf (1999):

> It seems to be a characteristic of the human mind that familiar concepts are abandoned only with the greatest reluctance, especially when a concrete picture of the phenomena has to be sacrificed.

That original quote actually refers to the decades-long struggle before get Maxwell's equations and modern electromagnetic theory were fully accepted. Indeed, some of the papers I read in the

course of preparing this note still referred to the "aether" well into the twentieth century, even though the Michelson and Morley experiment of 1887 failed to detect it, and Einstein's special relativity of 1905 showed there was no need for it. Max Planck, who had similar problems with the acceptance of his quantum ideas, said (Planck,1906b) that

> A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.

This is sometimes paraphrased as "Science advances one funeral at a time."

A similar struggle is underway, also with Mishchenko on the front line of battle, to correct the misuse of the word "photon" and the misconception of light as consisting of little particles zooming around in space. But that is another story, best told elsewhere. (Meanwhile, for more on that and a "History of Light" see http://www.oceanopticsbook.info/view/light_and_radiometry/level_2/the_nature_of_light)

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