<< Previous Chapter

Chapter 7

See Note

Invariant Imbedding Methods: Introduction

For the plane-parallel geometry found in most hydrologic optics problems, there exist solution methods that are vastly more efficient than Monte Carlo simulation. We now begin the development of one of these analytical (meaning deterministic or non-statistical) methods for solving the RTE in one spatial dimension. Many such methods are available; they go by names such as discrete ordinates methods, spherical harmonics methods, iterative methods, matrix methods, and invariant imbedding methods. Van de Hulst (1980) gives an excellent descriptive summary of the available solution methods, including the strengths and weaknesses of each. Some analytical methods are of great power and considerable generality. Others were developed for specific problems (such as Rayleigh scattering) and have found little or no application in hydrologic optics. Kattawar (1991) has compiled 43 original papers on solution methods applicable to the plane-parallel geometry of interest here. His book is a good place to survey the richness of mathematical methods that has been brought to bear on solving the RTE.

Our emphases in this book is on the development of solution methods arising from *invariant imbedding theory*, a branch of applied mathematics. Looking ahead, we will learn that these invariant imbedding methods have the following desirable characteristics:

- They are applicable to the solution of the RTE (5.23), including internal sources, depth-dependent IOP's, arbitrary incident radiances, wind-blown air-water surfaces, and a finite or infinite-depth bottom. The only restriction is that of plane-parallel geometry[1].

---

[1]Invariant imbedding methods can be formulated for other one-parameter geometries as well. Examples of their use can be found in problems with spherical or cylindrical symmetry in which quantities depend only on the radial coordinate.

- All quantities are computed with the same accuracy. In particular, there is no statistical noise in the numerical results.

- The methods are mathematically elegant and provide deep insights into the internal structure of radiative transfer theory. Many profound relationships are revealed among the building blocks of the theory.

- The methods are computationally efficient. The solution algorithms are fast and numerically stable. Moreover, computation time is a linear function of optical depth. Thus computing a radiance distribution from the surface to $\zeta = 10$ requires only twice the computer time as the same computation carried to $\zeta = 5$.

The price one pays for the above benefits is mathematical complexity. Monte Carlo methods are easily understood in terms of fundamental radiative processes and elementary probability theory. Invariant imbedding methods require a considerable amount of mathematical development in going from the RTE to its solution, and the associated computer programming is much more tedious.

Because of the mathematical sophistication required for a direct attack on the integro-differential RTE (5.23), we choose to introduce the essential aspects of invariant imbedding theory via its application to the much simpler two-flow irradiance equations [see Supplementary Note 11]. The two-flow equations, with their two unknowns E_d and E_u, are compactly represented in terms of 1×2 vectors and 2×2 matrices. After the manipulation of these quantities is mastered, it will be easy to make the transition to the radiance case. The radiance equations will have N unknowns, namely the quad-averaged radiances in N quads, and representations in terms of $1 \times N$ vectors and $N \times N$ matrices. The *form* of the radiance matrix equations, however, will be identical to the matrix equations developed here for irradiances. This chapter is concerned with developing invariant imbedding methods for solving the two-flow equations; the extension of these methods to the solution of the RTE is made in Chapter 8. Chapter 9, which discusses the special case of homogeneous water, will also make considerable use of the formulas developed here.

It is interesting to note that Monte Carlo and invariant imbedding methods are almost the same age. Monte Carlo methods were systematically developed during the second world war by scientists and mathematicians such as Fermi, von Neumann and Ulam. Their interest was in neutron diffusion problems associated with the development of the atomic bomb. The basic ideas of invariant imbedding can be traced back to an insight by the astrophysicist Ambarzumian (1943). The subsequent development of the theory and its application to radiative transfer problems was made by Chandrasekhar, Bellman, Preisendorfer, and others. The fullest exposition of invariant imbedding as applied to hydrologic optics is seen in Preisendorfer's six-volume treatise *Hydrologic Optics* (Preisendorfer, 1976).

One of the primary goals of the present book is to extract the essence of *Hydrologic Optics*, without the mathematical rigor and generality that characterize Preisendorfer's books. The path to this goal is not easy, but it can be taken one step at time. Let us begin.

7.1 The Two-Flow Equations as a Mathematics Problem

In Section 5.10 we obtained several equivalent forms of the two-flow equations for upward and downward plane irradiance. The most compact form was

$$\frac{dE_d(z)}{dz} = E_d(z)\,\tau_{dd}(z) + E_u(z)\,\rho_{ud}(z) + E^S_{od}(z)$$

$$-\frac{dE_u(z)}{dz} = E_u(z)\,\tau_{uu}(z) + E_d(z)\,\rho_{du}(z) + E^S_{ou}(z)\,.$$

We omit the wavelength argument λ for brevity, but we show the depth argument z to emphasize that all quantities can vary with depth.

We also saw that knowledge of the water IOP's, source functions, and boundary conditions is *insufficient* for solving these equations. We must provide additional information in order to determine the four τ's and ρ's, which are AOP's. For our present purposes, we assume that the needed additional information has been obtained either from measurement or from theoretical considerations, so that *the four* τ*'s and* ρ*'s are considered known,* as are the internal source functions E_{od}^{S} and E_{ou}^{S}.

The boundary conditions that are to be satisfied at the air-water surface $S[a,w]$ are given by Eqs. (4.5) and (4.6):

$$E_u(a) = E_u(w)\,t(w,a) + E_d(a)\,r(a,w)$$

$$E_d(w) = E_u(w)\,r(w,a) + E_d(a)\,t(a,w)\,.$$

We assume that the air-incident irradiance $E_d(a)$ is known; it is easily computed if the sky radiance distribution $L(a;\hat{\xi})$, $\hat{\xi} \in \Xi_d$, is given. Likewise, we assume that the four surface irradiance transfer functions (which are also AOP's) are known. They can be computed as in Chapter 4, given the sky radiance distribution and sea state. Note that the water-leaving irradiance $E_u(a)$ is unknown.

For concreteness, let us assume that the maximum depth of interest $z = m$ is an opaque Lambertian surface, as would be the case for a sandy or muddy bottom. Equation (4.83) then gives the boundary condition to be satisfied at $z = m$:

$$E_u(m) = E_d(m)\, r(m,b) = E_d(m)\, R\,,$$

where R is the known irradiance reflectance of the bottom boundary layer $S[m,b]$.

Table 7.1 collects the above equations and highlights the known quantities. It is now easy to state the mathematics problem at hand: find $E_d(z)$ and $E_u(z)$, $a \leq z \leq m$, such that each of Eqs. (7.1)-(7.5) is simultaneously satisfied. Actually finding these solution irradiances is less easy.

We note first that Eqs. (7.1)-(7.5) constitute a *two-point boundary value problem.* This means that the differential equations (7.3) and (7.4) must satisfy boundary conditions at two different depths, namely at the water surface and at the bottom. Two-point problems are generally much harder to solve than are *initial value problems*, which must satisfy a boundary condition at only one point. Suppose for the moment that we also know $E_u(a)$ and that the water is infinitely deep. Then we could solve Eq. (7.1) for $E_u(w)$, after which we could get $E_d(w)$ from Eq. (7.2). Then we could integrate Eqs. (7.3) and (7.4) downward starting with the known initial values $E_d(w)$ and $E_u(w)$, and thereby obtain $E_d(z)$ and $E_u(z)$ throughout the water column. But, alas, we do not know $E_u(a)$, and we do have to satisfy Eq. (7.5). *We can at least console ourselves with the observation that our differential equations are linear, for it is linearity that opens the door for invariant imbedding methods.*

7.2 Solution Algorithm for a Two-flow Problem

Although we cannot solve Eqs. (7.1)-(7.5) via a straightforward integration of the two-flow equations, we can however formulate an equivalent problem, which does yield to solution as an initial value problem. Since we seldom get something for nothing, we should not be surprised that we must add something to Eqs. (7.1)-(7.5) in order to effect the desired

Table 7.1. The two-flow irradiance equations and associated boundary conditions. The underlined quantities are assumed known.

water layer	equations to be satisfied	equation number
	$E_u(a) = E_u(w)\,\underline{t(w,a)} + \underline{E_d(a)}\,\underline{r(a,w)}$	
$S[a,w]$		(7.1)
	$E_d(w) = E_u(w)\,\underline{r(w,a)} + \underline{E_d(a)}\,\underline{t(a,w)}$	
		(7.2)
	$\dfrac{dE_d(z)}{dz} = E_d(z)\,\underline{\tau_{dd}(z)} + E_u(z)\,\underline{\rho_{ud}(z)} + \underline{E_{od}^{S}(z)}$	(7.3)
$S[w,m]$		
	$-\dfrac{dE_u(z)}{dz} = E_u(z)\,\underline{\tau_{uu}(z)} + E_d(z)\,\underline{\rho_{du}(z)} + \underline{E_{ou}^{S}(z)}$	(7.4)
		(7.5)
$S[m,b]$	$E_u(m) = E_d(m)\,\underline{r(m,b)}$	

transformation. That "something" is provided by our old friend, the interaction principle.

A global interaction principle

The interaction principle provided us with the boundary conditions at the water surface and bottom; recall Secs. 4.1 and 4.11. However, we have not yet exploited this principle *within* the water body $S[w,m]$. To see how this is done, suppose that we are at an arbitrary depth z within the water body, $w \le z \le m$. For an opaque bottom, the slab of water $S[z,b]$ is irradiated only by an *incident irradiance* $E_d(z)$; there is no irradiance $E_u(b)$ coming up from below the bottom boundary. Likewise the slab has a *response irradiance* $E_u(z)$, but $E_d(b) = 0$ for an opaque bottom. The internal sources $E_{ou}^{S}(z)$ and $E_{od}^{S}(z)$ within $S[z,m]$ give rise to some contribution $E_u^{t}(m,z)$ to the total response irradiance; $E_u^{t}(m,z)$ is independent of the incident irradiance $E_d(z)$. By assumption, there are no sources within the lower boundary $S[m,b]$.

The interaction principle applied to $S[z,b]$ asserts the existence of a *standard reflectance function* $R(z,b)$ and a *source-induced upward irradiance* $E_u^{t}(b,z)$ such that the total upward response irradiance at level z is given by

$$E_u(z) = E_d(z)\,R(z,b) + E_u^t(b,z)\,. \tag{7.6}$$

Equation (7.6) is the *global interaction principle* for the present problem.

The standard reflectance $R(z,b)$ is just the irradiance reflectance of everything – namely the water plus the bottom boundary – below depth z. The quantity $E_u^t(b,z)$ is the upwelling irradiance at z owing to the cumulative contribution of all sources between the bottom at b and level z. Beginning in Section 7.6, the superscript "t" will remind us that this source-induced irradiance is associated with what will be called the "transport" solution of the two-flow equations. A confusingly similar quantity will arise in our development in Section 7.5 of the so-called "fundamental" solution; that quantity will be given an "f" superscript.

But just as was the case with the air-water surface boundary conditions, the interaction principle is of no help unless we can find a way to compute $R(z,b)$ and $E_u^t(b,z)$. Fortunately, this is easily done.

Differential equations for $R(z,b)$ and $E_u^t(b,z)$

Let us differentiate the interaction principle (7.6) with respect to z:

$$\frac{dE_u(z)}{dz} = E_d(z)\,\frac{dR(z,b)}{dz} + \frac{dE_d(z)}{dz}\,R(z,b) + \frac{dE_u^t(b,z)}{dz}\,. \tag{7.7}$$

We can now use the two-flow equations (7.3) and (7.4) to replace dE_u/dz and dE_d/dz in Eq. (7.7). The result is

$$-E_u\,\tau_{uu} - E_d\,\rho_{du} - E_{ou}^S =$$
$$E_d\,\frac{dR}{dz} + \left[E_d\,\tau_{dd} + E_u\,\rho_{ud} + E_{od}^S\right]R + \frac{dE_u^t}{dz}\,,$$

where we have omitted the depth arguments for brevity. We can now use Eq. (7.6) to replace $E_u(z)$, which gives

$$-\left(E_d R + E_u^t\right)\tau_{uu} - E_d\,\rho_{du} - E_{ou}^S =$$
$$E_d\,\frac{dR}{dz} + \left[E_d\,\tau_{dd} + \left(E_d R + E_u^t\right)\rho_{ud} + E_{od}^S\right]R + \frac{dE_u^t}{dz}\,.$$

Let us now arrange this equation into two groups of terms, one with $E_d(z)$ as a coefficient and one containing only source-related terms. The result is

$$E_{\rm d}\left[-\frac{dR}{dz} - R\,\tau_{\rm uu} - \rho_{\rm du} - \tau_{\rm dd}R - R\,\rho_{\rm ud}R\right] + \left[-\frac{dE_{\rm u}^{\rm t}}{dz} - E_{\rm u}^{\rm t}\tau_{\rm uu} - E_{\rm ou}^{\rm S} - E_{\rm u}^{\rm t}\rho_{\rm ud}R - E_{\rm od}^{\rm S}R\right] = 0\,. \tag{7.8}$$

We now observe that *because the irradiance* $E_{\rm d}(z)$ *incident on the slab* $S[z,b]$ *is completely arbitrary and independent of the internal sources, the two quantities in brackets in Eq.* (7.8) *must individually be zero*. [Consider the analogous simple equations $5x + y = 0$ and $10x + y = 0$, which have only the simultaneous solution $x = 0$ and $y = 0$. Here the numbers 5 and 10 play the role of different values of $E_{\rm d}(z)$.] We thus obtain two differential equations for $R(z,b)$ and $E_{\rm u}^{\rm t}(b,z)$:

$$-\frac{dR(z,b)}{dz} = \rho_{\rm du}(z) + R(z,b)\,\tau_{\rm uu}(z) + \tau_{\rm dd}(z)\,R(z,b) + R(z,b)\,\rho_{\rm ud}(z)\,R(z,b) \tag{7.9}$$

$$-\frac{dE_{\rm u}^{\rm t}(b,z)}{dz} = E_{\rm u}^{\rm t}(b,z)\left[\tau_{\rm uu}(z) + \rho_{\rm ud}(z)\,R(z,b)\right] + E_{\rm ou}^{\rm S}(z) + E_{\rm od}^{\rm S}(z)\,R(z,b)\,. \tag{7.10}$$

These equations hold for all z in $w \le z \le m$. Note that Eq. (7.9) is *nonlinear*, owing to the presence of the $R\rho_{\rm ud}$R term.

Now when $z = m$, the depth of the opaque bottom, we have

$$R(m,b) = R \tag{7.11}$$

$$E_{\rm u}^{\rm t}(b,m) = 0\,. \tag{7.12}$$

The first equation follows because $R(z,b)$ for $z = m$ is the reflectance of everything below depth m, which is just the known reflectance R of the opaque Lambertian bottom. $E_{\rm u}^{\rm t}(b,z) = 0$ for $z = m$ because there are no internal sources within the opaque bottom $S[m,b]$. Note that Eqs. (7.11) and (7.12), when substituted into Eq. (7.6) evaluated at $z = m$, reduce Eq. (7.6) to the bottom boundary condition (7.5).

The τ's, ρ's and internal sources $E_{\rm ou}^{\rm S}$ and $E_{\rm od}^{\rm S}$ appearing in Eqs. (7.9) and (7.10) are all known throughout $S[w,m]$. We can therefore simultaneously integrate this pair of equations in an "upward sweep" from depth $z = m$ to depth $z = w$, beginning with the initial conditions (7.11) and (7.12) at $z = m$. The values of $R(z,b)$ and $E_{\rm u}^{\rm t}(b,z)$ just below the air-water surface are denoted by $R(w,b)$ and $E_{\rm u}^{\rm t}(b,w)$, respectively.

The integrations just described constitute the first step of our solution procedure. *Note that the upward integrations of Eqs. (7.9) and (7.10), beginning with the initial values (7.11) and (7.12), incorporate the lower boundary conditions into what will eventually be the solution irradiances.* We now show how to incorporate the upper boundary conditions into the solution.

Incorporation of the surface boundary conditions

The global interaction principle (7.6) holds for any z value within the water body; in particular it holds for $z = w$. We can use this observation to eliminate $E_u(w)$ from boundary condition (7.2), thereby obtaining an equation for $E_d(w)$. Thus Eq. (7.2) becomes

$$E_d(w) = \left[E_d(w)\, R(w,b) + E_u^t(b,w) \right] r(w,a) + E_d(a)\, t(a,w) \,.$$

Recall that $R(w,b)$ and $E_u^t(b,w)$ are now known from the integration of Eqs. (7.9) and (7.10). Solving this last equation for $E_d(w)$ yields

$$E_d(w) = E_d(a)\, \mathbf{T}(a,w,b) + \mathbf{E}(a,w,b) \,, \tag{7.13}$$

where

$$\mathbf{T}(a,w,b) \equiv t(a,w) \left[1 - R(w,b)\, r(w,a) \right]^{-1} \tag{7.14}$$

and

$$\mathbf{E}(a,w,b) \equiv E_u^t(b,w)\, r(w,a) \left[1 - R(w,b)\, r(w,a) \right]^{-1} . \tag{7.15}$$

We now know the downward irradiance just below the air-water surface.

We can obtain $E_u(w)$ in a similar fashion. Substituting Eq. (7.13) for the left-hand-side of boundary condition (7.2) gives

$$E_d(a)\, \mathbf{T}(a,w,b) + \mathbf{E}(a,w,b) = E_u(w)\, r(w,a) + E_d(a)\, t(a,w) \,,$$

which involves only $E_u(w)$ and known quantities. Solving for $E_u(w)$ and writing out $\mathbf{T}(a,w,b)$ and $\mathbf{E}(a,w,b)$ as in Eqs. (7.14) and (7.15) gives

$$\begin{aligned} E_u(w) &= E_d(a) \left\{ t(a,w) \left[1 - R(w,b)\, r(w,a) \right]^{-1} - t(a,w) \right\} r^{-1}(w,a) \\ &\quad + E_u^t(b,w)\, r(w,a) \left[1 - R(w,b)\, r(w,a) \right]^{-1} r^{-1}(w,a) \,. \end{aligned}$$

The first term on the right-hand-side can be simplified by letting $x =$

$R(w,b)r(w,a)$ and recalling that

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots .$$

[Note that $0 \le x < 1$ because both $R(w,b)$ and $r(w,a)$ are reflectances, which are restricted to the interval 0 to 1 on physical grounds.] The resulting equation for $E_u(w)$ is

$$E_u(w) = E_d(a)\,\mathbf{R}(a,w,b) + \mathbf{E}(b,w,a) \tag{7.16}$$

where

$$\mathbf{R}(a,w,b) \equiv t(a,w)\left[1 - R(w,b)\,r(w,a)\right]^{-1} R(w,b) \tag{7.17}$$

and

$$\mathbf{E}(b,w,a) \equiv E_u^{t}(b,w)\left[1 - r(w,a)\,R(w,b)\right]^{-1} . \tag{7.18}$$

Clearly, the values of $E_d(w)$ and $E_u(w)$ just obtained incorporate information about the nature of the air-water surface itself (via the r and t factors) as well as about the water body and bottom boundary [the $R(w,b)$ term], and the internal sources [the $E_u^t(b,w)$ term].

We are now able to integrate the two-flow equations (7.3) and (7.4) as an initial value problem, starting with the $E_d(w)$ and $E_u(w)$ values just computed and integrating downward from $z = w$ to the bottom at depth $z = m$. This integration yields the irradiances throughout the water body $S[w,m]$. In practice, we would save the values of $E_d(z)$ and $E_u(z)$ at whatever specific depths z_1, z_2, ... were of interest to us in a particular problem.

The only remaining unknown is the upward irradiance leaving the air-water surface. $E_u(a)$ is easily obtained from boundary condition (7.1), because we now know $E_u(w)$. *We have now completely solved the mathematics problem posed in the previous section.*

Recapitulation and interpretation

Because the preceding development was somewhat abstract, it is worthwhile to summarize what we have done, and to provide a physical interpretation for the mathematical operations. The solution of Eqs. (7.1)-(7.5) involves five steps:

(i) Integrate Eqs. (7.9) and (7.10) in an upward sweep from $z = m$ to $z = w$, starting with the initial conditions (7.11) and (7.12), to obtain $R(w,b)$ and $E_u^t(b,w)$.

(ii) Compute $\mathbf{T}(a,w,b)$ and $\mathbf{E}(a,w,b)$, and $\mathbf{R}(a,w,b)$ and $\mathbf{E}(b,w,a)$, by Eqs. (7.14) and (7.15), and (7.17) and (7.18), respectively.

(iii) Compute $E_d(w)$ and $E_u(w)$ by Eqs. (7.13) and (7.16), respectively.

(iv) Integrate the two-flow Eqs. (7.3) and (7.4) in a downward sweep from $z = w$ to $z = m$, starting with the initial values obtained in step (iii), and saving the results at all desired depths $z,\ w \le z \le m$.

(v) Compute $E_u(a)$ from Eq. (7.1).

Each step of the solution algorithm has a physical interpretation. The integrations in step (i) represent the "construction" of the water body $S[w,m]$ by starting with the bottom boundary and adding infinitesimal layers of water. Equations (7.9) and (7.10) govern how the reflectance and internal source properties of the water body evolve as more and more layers of water are added.

$\mathbf{T}(a,w,b)$ is called the *complete downward transmittance of* $S[a,b]$ *at level* w. It has a most important interpretation. We can understand its physical significance by recalling its use in Eq. (7.13). For the moment, assume that there are no internal sources, so that $E_u^t(b,w) = 0$, and hence $\mathbf{E}(a,w,b)$. Then Eq. (7.13) reads

$$\begin{aligned} E_d(w) &= E_d(a)\,\mathbf{T}(a,w,b) \\ &= E_d(a)\,t(a,w)\left[1 - R(w,b)\,r(w,a)\right]^{-1} \\ &= E_d(a)\,t(a,w)\left[1 + R(w,b)\,r(w,a)\right. \\ &\qquad\qquad \left. + R(w,b)\,r(w,a)\,R(w,b)\,r(w,a) + \cdots\right], \end{aligned} \tag{7.19}$$

where we have expanded $(1 - x)^{-1}$ as before. The first term in the series expansion on the right-hand side of Eq. (7.19) represents a contribution to $E_d(w)$ from $E_d(a)$ being transmitted [by $t(a,w)$] through the air-water surface. The second term in this series represents the part of $E_d(a)$ that has been transmitted through the surface, reflected back upward by everything below level w [the $R(w,b)$ term], and then reflected back downward again by the air-water surface [the $r(w,a)$ term]. The subsequent terms of the series represent successive inter-reflections of this nature. Figure 7.1 illustrates the first three terms on the right-hand side of Eq. (7.19).

The *complete downward reflectance* $\mathbf{R}(a,w,b)$ *of* $S[a,b]$ *at level* w has a similar interpretation. As is seen in Eq. (7.17), each of the inter-reflection terms just described for $\mathbf{T}(a,w,b)$ is now followed by one more reflection

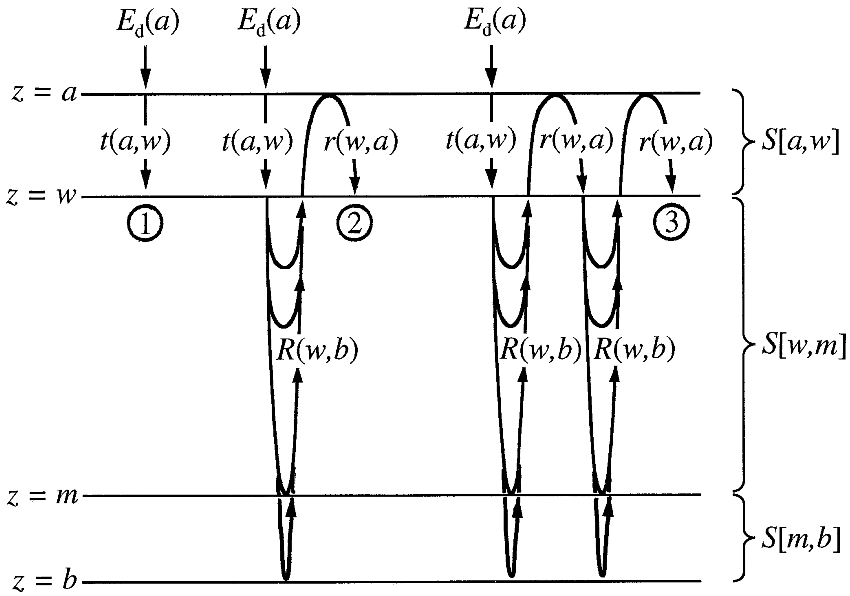

Fig. 7.1. Illustration of the physical significance of the complete transmittance $\mathbf{T}(a,w,b)$. The circled numbers correspond to the first three terms of the series expansion in Eq. (7.19)

$R(w,b)$, which converts downwelling irradiance into an upwelling irradiance contribution to $E_u(w)$ in Eq. (7.16).

In a similar manner, the *complete source-induced downward irradiance* $\mathbf{E}(a,w,b)$ defined in Eq. (7.15) takes the upwelling irradiance $E_u^{\,i}(b,w)$ generated at depth w by internal sources between the bottom at b and depth w, reflects it back downward by the air-water surface, and carries out an infinite series of inter-reflections between the water body plus bottom, $S[w,b]$, and the surface $S[a,w]$. The *complete source-induced upward irradiance* $\mathbf{E}(b,w,a)$ has a similar interpretation.

The true importance of the complete operators[1] $\mathbf{T}$, $\mathbf{R}$, and $\mathbf{E}$ is this: *they account for all orders of multiple scattering within the entire medium* $S[a,b]$. The inclusion of multiple scattering effects in radiative transfer problems is often difficult. There are many techniques for obtaining

---

[1]The standard notation for the complete operators $\mathbf{T}$, $\mathbf{R}$, and $\mathbf{E}$ (as seen in *H.O.*, for example) uses script letters instead of bold face. Unfortunately, a script font was not available on the author's wordprocessor.

approximate solutions of the RTE that ignore multiple scattering effects. The term $[1 - R(w,b)r(w,a)]^{-1}$, which appeared so naturally in the preceding development, represents the *algebraic sum of an infinite number of inter-reflections, which is equivalent to an infinite number of multiple scatterings by the photons within the medium.* This last statement is made here without proof, but it should at least seem plausible to the reader.

Preisendorfer shows that the above algebraic solution procedure is completely equivalent to the so-called *natural solution procedure*, or the *scattering-order solution procedure*, in which the radiance (in general) is expanded in a series of terms representing the contributions of successive orders of multiple scattering. In these solution procedures, the calculations are terminated after a certain number of multiple scattering orders has been accounted for. These matters are discussed in detail in *H.O. III*, Chapter 5 and in *H.O. IV*. We also note that Monte Carlo methods effectively account for multiple scattering if the photon packets are followed until they contain a negligible amount of energy.

It should by now be obvious that we are on the trail of some very powerful mathematical concepts. *We have seen*, albeit in a very simple setting, *how to transform a linear, two-point boundary value problem into a nonlinear initial value problem. Such a transformation is the hallmark of invariant imbedding theory.* Although non-linear differential equations are wisely avoided by those who work only with a pencil and paper, the numerical solution of our particular equations poses no problems. Moreover, *we have obtained a solution that accounts for all orders of multiple scattering, is free of statistical noise, and produces intermediate quantities that are easily interpreted.*

It is certainly true that for the irradiance problem just discussed, we could simply measure E_d and E_u just below the water surface, and then begin with solution step (iv) above. But our goal here is to introduce a solution method. Looking ahead to Chapter 8, we will find that these same ideas are applicable to the solution of the RTE. There, we can reasonably expect to know the incident sky radiance distribution, but certainly not the full radiance distribution just below the surface. Moreover, we will find that the corresponding τ's and ρ's are functions only of the inherent optical properties of the water body, so that no assumptions about their values will be needed in order to perform the integrations corresponding to steps (i) and (iv) above. We may also anticipate that the quantities corresponding to $R(z,b)$, $r(a,w)$, etc. will all be matrices, because they will describe the reflectance of radiance in many different directions, not just from the downward to the upward hemispheres. However, there is much more to be learned at the irradiance level before tackling the radiances.

7.3 Transport and Fundamental Operators for the Air-Water Surface

The surface boundary conditions

$$E_u(a) = E_u(w)\,t(w,a) + E_d(a)\,r(a,w)$$
$$E_d(w) = E_u(w)\,r(w,a) + E_d(a)\,t(a,w)$$

can be written in matrix form as

$$\left[E_u(a)\,,E_d(w)\right] = \left[E_u(w)\,,E_d(a)\right]\begin{bmatrix} t(w,a) & r(w,a) \\ r(a,w) & t(a,w)\end{bmatrix} \tag{7.20}$$
$$\equiv \left[E_u(w)\,,E_d(a)\right]\underline{M}(a,w)\,.$$

The 2×2 matrix $\underline{M}(a,w)$ is called the *transport operator* (or *transport matrix*) for slab $S[a,w]$. $\underline{M}(a,w)$ is so named because its elements show how irradiance is transported (i.e. transmitted and reflected) back and forth by the slab $S[a,w]$. Note also that $\underline{M}(a,w)$ transforms the irradiances *incident* on $S[a,w]$ from above and below, $E_d(a)$ and $E_u(w)$, into the *response* irradiances $E_u(a)$ and $E_d(w)$, which are leaving $S[a,w]$.

We can also solve the surface boundary conditions for $E_u(w)$ and $E_d(w)$ in terms of $E_u(a)$ and $E_d(a)$. The result is

$$E_u(w) = E_u(a)\,t^{-1}(w,a) - E_d(a)\,r(a,w)\,t^{-1}(w,a)$$
$$E_d(w) = E_u(a)\,t^{-1}(w,a)\,r(w,a) + E_d(a)\left[t(a,w) - r(a,w)\,t^{-1}(w,a)\,r(w,a)\right].$$

Placing these equations in matrix form gives

$$\left[E_u(w)\,,E_d(w)\right] = \left[E_u(a)\,,E_d(a)\right]\begin{bmatrix} t^{-1}(w,a) & t^{-1}(w,a)\,r(w,a) \\ -r(a,w)\,t^{-1}(w,a) & t(a,w) - r(a,w)\,t^{-1}(w,a)\,r(w,a)\end{bmatrix} \tag{7.21a}$$

$$\equiv \left[E_u(a)\,,E_d(a)\right]\begin{bmatrix} \mathbf{m}_{--}(a,w) & \mathbf{m}_{-+}(a,w) \\ \mathbf{m}_{+-}(a,w) & \mathbf{m}_{++}(a,w)\end{bmatrix} \tag{7.21b}$$

$$\equiv \left[E_u(a)\,,E_d(a)\right]\underline{\mathbf{M}}(a,w)\,. \tag{7.21c}$$

The 2×2 matrix $\underline{\mathbf{M}}(a,w)$ is called the *fundamental operator* (or *fundamental matrix*) for $S[a,w]$. Note that $\underline{\mathbf{M}}(a,w)$ transforms the upward and downward irradiances at *one depth* ($z = a$) into the irradiances at *another depth* ($z = w$). The "+" and "−" subscripts on the elements of $\underline{\mathbf{M}}(a,w)$ remind us of which elements transform downward irradiance (the + or Ξ_d direction) into upward irradiance (the − or Ξ_u direction), and so on[1].

Equations (7.21a) and (7.21b) show how to obtain the elements of $\underline{\mathbf{M}}(a,w)$ from those of $\underline{M}(a,w)$. The inverse transformation is easily obtained by writing out Eq. (7.21b) as two equations, and then solving them for $E_u(a)$ and $E_d(w)$ in terms of $E_u(w)$ and $E_d(a)$. One can then make the identification

$$\begin{aligned}\underline{M}(a,w) &= \begin{bmatrix} t(w,a) & r(w,a) \\ r(a,w) & t(a,w) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{m}_{--}^{-1}(a,w) & \mathbf{m}_{--}^{-1}(a,w)\,\mathbf{m}_{-+}(a,w) \\ -\mathbf{m}_{+-}(a,w)\,\mathbf{m}_{--}^{-1}(a,w) & \mathbf{m}_{++}(a,w) - \mathbf{m}_{+-}(a,w)\,\mathbf{m}_{--}^{-1}(a,w)\,\mathbf{m}_{-+}(a,w) \end{bmatrix}.\end{aligned} \tag{7.22}$$

The similar structures of the matrices in Eqs. (7.21) and (7.22) should be noted. The operators $\underline{M}(a,w)$ and $\underline{\mathbf{M}}(a,w)$, and their generalizations, will play central roles in our subsequent developments.

7.4 The Fundamental Solution for Source-free Water Bodies

Let us now re-examine step (iv) of the solution procedure of Section 7.2. This was the step in which we integrated the two-flow equations (7.3) and (7.4) downward from depth w. The initial values for the integration were the irradiances $E_u(w)$ and $E_d(w)$ just below the surface. These values were determined by the combined effects of the incident irradiance $E_d(a)$, the air-water and bottom boundaries, the internal sources, and the τ's and

---

[1]We have chosen downward as the positive (+) direction. Preisendorfer (1976) always chose upward as the positive direction; therefore his +(−) is our −(+). When comparing the present results with the corresponding development in *H.O.* or in other works with positive upward, it is necessary to interchange plus and minus signs on all such quantities. Also, *H.O.* uses a script M for fundamental operators.

ρ's of the entire water body. The results of the integration were the irradiances $E_u(z)$ and $E_d(z)$ at any depth within the water body $S[w,m]$.

We now pose a question. Is it possible to integrate the two-flow equations using "general" initial values, thereby obtaining a "general" set of irradiances, which can then be used to generate the specific solution irradiances for any particular values of the initial conditions? If the answer is yes (and it is), then we can integrate the two-flow equations once for any given water body (i.e. for any given set of τ's and ρ's), and then "apply" different boundary conditions to the general solution in order to obtain solutions for the particular set of boundary conditions of interest. The details of this process follow.

For simplicity, let us first consider the case of a source-free water body: $E_{od}^{S}(z) = E_{ou}^{S}(z) = 0$. Now let

$$E_u^{(1)}(w) \equiv 1 \quad \text{and} \quad E_d^{(0)}(w) \equiv 0 \tag{7.23}$$

be a pair of *dimensionless initial values* at level w. We next integrate the source-free two-flow equations

$$\begin{aligned} \frac{dE_d(z)}{dz} &= E_d(z)\,\tau_{dd}(z) + E_u(z)\,\rho_{ud}(z) \\ -\frac{dE_u(z)}{dz} &= E_u(z)\,\tau_{uu}(z) + E_d(z)\,\rho_{du}(z)\,. \end{aligned} \tag{7.24}$$

in a downward sweep from level w to any depth z, beginning with the initial values of Eq. (7.23). Call the dimensionless results of this integration $E_u^{(1)}(z)$ and $E_d^{(0)}(z)$. Now repeat this integration of Eqs. (7.24) beginning with

$$E_u^{(0)}(w) = 0 \quad \text{and} \quad E_d^{(1)}(w) = 1\,,$$

and call the results $E_u^{(0)}(z)$ and $E_d^{(1)}(z)$.

It soon will be convenient to write these results in matrix form, so define

$$\begin{aligned} \mathbf{M}_{--}(w,z) &\equiv E_u^{(1)}(z) \\ \mathbf{M}_{-+}(w,z) &\equiv E_d^{(0)}(z) \\ \mathbf{M}_{+-}(w,z) &\equiv E_u^{(0)}(z) \\ \mathbf{M}_{++}(w,z) &\equiv E_d^{(1)}(z)\,. \end{aligned}$$

By virtue of their construction, $\mathbf{M}_{-+}(w,z)$ and $\mathbf{M}_{++}(w,z)$ satisfy

$$\frac{d}{dz}\mathbf{M}_{\mp+}(w,z) = \mathbf{M}_{\mp+}(w,z)\,\tau_{++}(z) + \mathbf{M}_{\mp-}(w,z)\,\rho_{-+}(z) \tag{7.26}$$

with the initial conditions

$$\mathbf{M}_{-+}(w,w) = 0 \quad \text{and} \quad \mathbf{M}_{++}(w,w) = 1\,. \tag{7.27}$$

Here we read upper signs together and lower signs together on the **M**'s. We have replaced the "d" and "u" subscripts on the τ's and ρ's with "+" and "−", respectively, for consistency with the **M**'s. Likewise $\mathbf{M}_{--}(w,z)$ and $\mathbf{M}_{+-}(w,z)$ satisfy

$$-\frac{d}{dz}\mathbf{M}_{\mp-}(w,z) = \mathbf{M}_{\mp-}(w,z)\,\tau_{--}(z) + \mathbf{M}_{\mp+}(w,z)\,\rho_{+-}(z) \tag{7.28}$$

with the initial conditions

$$\mathbf{M}_{--}(w,w) = 1 \quad \text{and} \quad \mathbf{M}_{+-}(w,w) = 0\,. \tag{7.29}$$

Equation (7.26) shows that $\mathbf{M}_{-+}$ and $\mathbf{M}_{++}$ behave differentially like E_d, and Eq. (7.28) shows that $\mathbf{M}_{--}$ and $\mathbf{M}_{+-}$ behave like E_u.

The two pairs of dimensionless functions $[\mathbf{M}_{--}(w,z), \mathbf{M}_{-+}(w,z)]$ and $[\mathbf{M}_{+-}(w,z), \mathbf{M}_{++}(w,z)]$, where w is held fixed and z varies, constitute the *fundamental solutions* of the source-free two-flow Eqs. (7.24). They are given this distinguished name because *any solution of Eq. (7.24) can be written as a linear combination of them.* The proof of this statement is as follows. The 1×2 vectors representing the initial conditions at $z = w$, namely

$$\left[\mathbf{M}_{--}(w,w), \mathbf{M}_{-+}(w,w)\right] = [1, 0]$$

and

$$\left[\mathbf{M}_{+-}(w,w), \mathbf{M}_{++}(w,w)\right] = [0, 1]\,,$$

are clearly linearly independent. It then follows from the theory of differential equations that the solution vectors of Eqs. (7.26) and (7.28) remain linearly independent for any depth z, if the τ's and ρ's are continuous functions of z (see Coddington and Levinson, 1955, pp 28 and 69). The linearly independent fundamental solutions at depth z therefore can serve as a basis for the representation of any arbitrary 1×2 solution vector. In particular, if $E_u(z)$ and $E_d(z)$ are the results of integrating Eqs. (7.24) starting with the *arbitrary* initial conditions $E_u(w)$ and $E_d(w)$, then

$$E_u(z) = E_u(w)\,\mathbf{M}_{--}(w,z) + E_d(w)\,\mathbf{M}_{+-}(w,z)$$

$$E_d(z) = E_u(w)\,\mathbf{M}_{-+}(w,z) + E_d(w)\,\mathbf{M}_{++}(w,z)\,.$$

Placing these equations in matrix form gives

$$\left[E_{\mathrm{u}}(z)\,,E_{\mathrm{d}}(z)\right] = \left[E_{\mathrm{u}}(w)\,,E_{\mathrm{d}}(w)\right]\begin{bmatrix}\mathbf{M}_{--}(w,z) & \mathbf{M}_{-+}(w,z)\\ \mathbf{M}_{+-}(w,z) & \mathbf{M}_{++}(w,z)\end{bmatrix} \tag{7.30}$$

$$\equiv \left[E_{\mathrm{u}}(w)\,,E_{\mathrm{d}}(w)\right]\underline{\mathbf{M}}(w,z)\,.$$

Note that $\underline{\mathbf{M}}(w,z)$, the fundamental operator for $S[w,z]$,, works exactly like the fundamental operator for the air-water surface, $\underline{\mathbf{M}}(a,w)$, which was defined in the previous section. $\underline{\mathbf{M}}(w,z)$ transforms the solution at depth w into the solution at any depth z within the water body. Only the manner in which the elements of $\underline{\mathbf{M}}(w,z)$ and $\underline{\mathbf{M}}(a,w)$ are computed is different in the two cases.

The mapping and group properties of the fundamental solution

We now point out two important properties of the fundamental solution. First, it can be associated with *any* pair of depths z_o and z in $S[w,m]$. We can repeat the integrations of Eqs. (7.26) and (7.27), starting at any depth $w \le z_o \le m$, with the initial conditions

$$\underline{\mathbf{M}}(z_o,z_o) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \underline{I}_2\,,$$

and integrating to any other depth z, which can be greater than or less than z_o . $\underline{I}_2$ denotes the 2×2 identity matrix. Equation (7.30) then has the more general form

$$\left[E_{\mathrm{u}}(z)\,,E_{\mathrm{d}}(z)\right] = \left[E_{\mathrm{u}}(z_o)\,,E_{\mathrm{d}}(z_o)\right]\underline{\mathbf{M}}(z_o,z)\,, \tag{7.31}$$

which is known as the *mapping property* of $\underline{\mathbf{M}}(w,z)$.

Second, by virtue of its construction via the integrations of Eqs. (7.26) and (7.28), $\underline{\mathbf{M}}(z_o,z)$ possesses the *group property*

$$\underline{\mathbf{M}}(z_o,z) = \underline{\mathbf{M}}(z_o,z_1)\,\underline{\mathbf{M}}(z_1,z)\,. \tag{7.32}$$

In constructing $\underline{\mathbf{M}}(z_1,z)$, we of course start the integration at depth z_1, using the initial conditions $\underline{\mathbf{M}}(z_1,z_1) = \underline{I}_2$. Letting $z_o = z$ in Eq. (7.32) gives

$$\underline{\mathbf{M}}(z,z) = \underline{I}_2 = \underline{\mathbf{M}}(z,z_1)\,\underline{\mathbf{M}}(z_1,z)\,,$$

which means that

$$\underline{\mathbf{M}}(z_1,z) = \underline{\mathbf{M}}^{-1}(z,z_1)\,. \tag{7.33}$$

The fundamental operator $\underline{\mathbf{M}}(z_1,z)$ therefore has a unique inverse that can map the solution at z back to z_1.

7.5 The Fundamental Solution Including Internal Sources

The preceding two sections have illustrated the conceptual power of the fundamental solutions. We now need to extend the above developments to include the effects of internal sources within the water body.

The two-flow equations in matrix form

The convenience of matrix notation should by now be obvious. We can write the general two-flow equations (7.3) and (7.4) in matrix form by defining

$$\underline{K}(z) \equiv \begin{bmatrix} -\tau_{uu}(z) & \rho_{ud}(z) \\ -\rho_{du}(z) & \tau_{dd}(z) \end{bmatrix}, \tag{7.34}$$

$$\underline{E}(z) \equiv \left[E_u(z)\,,E_d(z)\right], \tag{7.35}$$

and

$$\underline{E}_o^S(z) \equiv \left[-E_{ou}^S(z)\,,E_{od}^S(z)\right]. \tag{7.36}$$

The minus sign on the $E_{ou}{}^S(z)$ component of $\underline{E}_o{}^S(z)$, like those seen in $\underline{K}(z)$, merely accounts for the sign in Eq. (7.4); it in no way implies a negative value for the irradiance itself. The two-flow equations now become

$$\frac{d}{dz}\underline{E}(z) = \underline{E}(z)\,\underline{K}(z) + \underline{E}_o^S(z)\,. \tag{7.37}$$

The 2×2 matrix $\underline{K}(z)$ is known as the *local transfer matrix* because it specifies the radiative transfer properties of the medium in terms of the local transmittances and local reflectances.

Using definition (7.34), Eqs. (7.26) and (7.28) have the compact form

$$\frac{d}{dz}\underline{\mathbf{M}}(w,z) = \underline{\mathbf{M}}(w,z)\,\underline{K}(z)\,, \tag{7.38}$$

where the initial conditions are now

$$\underline{\mathbf{M}}(w,w) = \underline{I}_2\,. \tag{7.39}$$

The solution of the source-free form of Eq. (7.37) is now expressible in terms of the fundamental operator as

$$\underline{E}(z) = \underline{E}(w)\,\underline{\mathbf{M}}(w,z)\,. \tag{7.40}$$

Incorporation of internal sources

We can deduce the required analytic form of the solution of Eq. (7.37) by physical reasoning. Consider an infinitesimally thin layer of water of thickness dz_o between depths z_o and $z \equiv z_o + dz_o$, which can be above or below z_o. The internal sources at level z_o generate an irradiance

$$\underline{E}_o^S(z_o)\,dz_o = \left[-E_{ou}^S(z_o)\,,\,E_{od}^S(z_o)\right]dz_o\,.$$

at level z. This internal-source irradiance vector acts just like the initial irradiance vector $\underline{E}(w) = [E_u(z_o), E_d(z_o)]$ in Eq. (7.31). Thus $\underline{E}_o^S(z_o)dz_o\underline{\mathbf{M}}(z_o,z)$ is the resultant irradiance vector at depth z induced by the internal sources at depth z_o. Adding up such contributions from all levels z' between any two arbitrary levels z_o and z gives

$$\begin{aligned}\underline{E}^f(z_o,z) &\equiv \int_{z_o}^{z}\underline{E}_o^S(z')\,\underline{\mathbf{M}}(z',z)\,dz' \\ &\equiv \left[E_u^f(z_o,z)\,,\,E_d^f(z_o,z)\right].\end{aligned} \tag{7.41}$$

The superscript "f" on $E_u^f(z_o,z)$ and $E_d^f(z_o,z)$, and on their vector form $\underline{E}^f(z_o,z)$, reminds us that these *fundamental source-induced irradiances* are associated with the fundamental operator.

We now can add the source-induced contribution (7.41) to the source-free solution (7.40) to generate the *general solution* of Eq. (7.37):

$$\underline{E}(z) = \underline{E}(z_o)\,\underline{\mathbf{M}}(z_o,z) + \underline{E}^f(z_o,z)\,. \tag{7.42}$$

We leave it as an exercise for the reader to verify that the vector $\underline{E}(z)$ of Eq.(7.42) does indeed satisfy the two-flow equations and the associated initial conditions. Equation (7.42) is valid for all z_o and z in $S[w,m]$.

Two special cases of Eq. (7.42) will be of use in the next section. The first is obtained by setting $z_o = w$, the depth just below the air-water surface. Then Eq. (7.42) can be expanded into

$$E_u(z) = E_u(w)\,\mathbf{M}_{--}(w,z) + E_d(w)\,\mathbf{M}_{+-}(w,z) + E_u^f(w,z) \tag{7.43}$$

$$E_d(z) = E_u(w)\,\mathbf{M}_{-+}(w,z) + E_d(w)\,\mathbf{M}_{++}(w,z) + E_d^f(w,z)\,. \tag{7.44}$$

The second case occurs when $z_o = m$, the maximum depth of interest. Equation (7.42) then has the same form as Eqs. (7.43) and (7.44), but with m replacing w:

$$E_u(z) = E_u(m)\,\mathbf{M}_{--}(m,z) + E_d(m)\,\mathbf{M}_{+-}(m,z) + E_u^f(m,z) \tag{7.45}$$

$$E_d(z) = E_u(m)\,\mathbf{M}_{-+}(m,z) + E_d(m)\,\mathbf{M}_{++}(m,z) + E_d^f(m,z)\,. \tag{7.46}$$

Historical notes

The fundamental-solution approach to radiative transfer theory as seen here is a direct application of the classical theory of differential equations; see, for example, Coddington and Levinson (1955). However, this approach was discovered independently by Preisendorfer after close analysis of early formulations of the interaction principle. For example, equations corresponding (at the radiance level) to Eqs. (7.32) and (7.40) were first presented in Preisendorfer (1961). Fundamental solutions for source-free media are treated in detail in *H.O. IV*. The extension of the theory to include internal sources was first presented in Preisendorfer and Mobley (1988).

The fundamental-solution formalism can be viewed as the "mathematical" approach to radiative transfer theory. The "physical" approach is given by the transport-solution formalism. The transport formalism is developed in *H.O. II*, Chapter 3, starting with the interaction principle. We next investigate how the transport-solution and the fundamental-solution treatments of radiative transfer theory are related.

7.6 The Transport Solution for Bare Slabs

The transport solution of the two-flow equations arises when we pose the following problem. As always, it is assumed that we know the local transmittances τ_{dd} and τ_{uu}, the local reflectances ρ_{du} and ρ_{ud}, and the internal sources $E_{ou}{}^{S}$ and $E_{od}{}^{S}$ throughout the water body $S[w,m]$. Suppose that we know the *incident irradiances* $E_d(w)$ *and* $E_u(z)$ *on the slab* $S[w,z]$. We wish to find the *response irradiances* $E_u(w)$ *and* $E_d(z)$ *leaving the slab*. This in turn requires us to determine four numbers, namely $T(z,w)$, $R(z,w)$, $T(w,z)$ and $R(w,z)$, and two irradiances $E_u{}^{t}(z,w)$ and $E_d{}^{t}(w,z)$, such that

$$E_u(w) = E_u(z)\,T(z,w) + E_d(w)\,R(w,z) + E_u^{t}(z,w) \tag{7.47}$$

$$E_d(z) = E_u(z)\,R(z,w) + E_d(w)\,T(w,z) + E_d^{t}(w,z)\,. \tag{7.48}$$

The R's and T's in Eqs. (7.47) and (7.48) are called the *standard reflectances* and *standard transmittances* for slab $S[w,z]$. We shall use the term *standard operators* to denote the standard reflectances and transmittances along with the *transport source-induced irradiances* $E_u{}^{t}$ and $E_d{}^{t}$.

The physical interpretation required of $T(z,w)$ is that of an upward transmittance that carries a part of the incident irradiance $E_u(z)$ upward through the slab to depth w. Likewise, $R(w,z)$ is a reflectance that shows how much of $E_d(w)$ is returned upward at level w by the slab $S[w,z]$. $E_u{}^{t}(z,w)$ is the upward irradiance at level w generated by the internal sources within the slab; the superscript "t" reminds us that this quantity is associated with the transport solution. Note in particular that the transport source-induced irradiance $E_u{}^{t}(z,w)$ is distinct from the fundamental source-induced irradiance $E_u{}^{f}(z,w)$, which is obtained from Eq. (7.41) by letting $z_o = z$ and $z = w$. The quantities in Eq. (7.48) have correspondingly simple interpretations.

The reader will surely note that Eqs. (7.47) and (7.48) strongly resemble equations that we have written down previously after invoking the interaction principle; recall, for example, the surface boundary conditions (7.1) and (7.2). We shall say more about this matter presently. For the moment, though, let us continue with the observation that the needed R, T, and $E^{\,t}$ functions of Eqs. (7.47) and (7.48) can be obtained immediately from the fundamental solutions, which we have just computed. Solving Eqs. (7.43) and (7.44) for $E_u(w)$ and $E_d(z)$ gives

$$E_u(w) = E_u(z)\,\mathbf{M}_{--}^{-1} - E_d(w)\,\mathbf{M}_{+-}\,\mathbf{M}_{--}^{-1} - E_u^f(w,z)\,\mathbf{M}_{--}^{-1} \tag{7.49}$$

$$\begin{aligned} E_d(z) &= E_u(z)\,\mathbf{M}_{--}^{-1}\,\mathbf{M}_{-+} + E_d(w)\left[\mathbf{M}_{++} - \mathbf{M}_{+-}\,\mathbf{M}_{--}^{-1}\,\mathbf{M}_{-+}\right] \\ &\quad + E_d^f(w,z) - E_u^f(w,z)\,\mathbf{M}_{--}^{-1}\,\mathbf{M}_{-+}\,, \end{aligned} \tag{7.50}$$

where each $\mathbf{M}_{\pm\pm}$ has depth arguments (w,z). Comparing these equations with Eqs. (7.47) and (7.48) shows that

$$\begin{bmatrix} T(z,w) & R(z,w) \\ R(w,z) & T(w,z) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{--}^{-1}(w,z) & \mathbf{M}_{--}^{-1}(w,z)\,\mathbf{M}_{-+}(w,z) \\ -\mathbf{M}_{+-}(w,z)\,\mathbf{M}_{--}^{-1}(w,z) & \mathbf{M}_{++}(w,z) - \mathbf{M}_{+-}(w,z)\,\mathbf{M}_{--}^{-1}(w,z)\,\mathbf{M}_{-+}(w,z) \end{bmatrix}, \tag{7.51}$$

and that

$$\left[E_u^t(z,w)\,,\,E_d^t(w,z)\right] = \left[-E_u^f(w,z)\,\mathbf{M}_{--}^{-1}(w,z)\,,\,E_d^f(w,z) - E_u^f(w,z)\,\mathbf{M}_{--}^{-1}(w,z)\,\mathbf{M}_{-+}(w,z)\right]. \tag{7.52}$$

Equations (7.39) and (7.41) show that if $z = w$, the last two equations reduce to

$$\begin{bmatrix} T(w,w) & R(w,w) \\ R(w,w) & T(w,w) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.53}$$

and

$$\left[E_u^t(w,w)\,,\,E_d^t(w,w)\right] = \left[0\,,0\right], \tag{7.54}$$

as we expect on physical grounds.

Proceeding in parallel with the above analysis, we can suppose that we know the incident irradiances at depths z and m, $E_d(z)$ and $E_u(m)$, respectively. The response irradiances for the slab $S[z,m]$ are then $E_u(z)$ and $E_d(m)$, and we can write

$$E_u(z) = E_u(m)\,T(m,z) + E_d(z)\,R(z,m) + E_u^t(m,z) \tag{7.55}$$

$$E_d(m) = E_u(m)\,R(m,z) + E_d(z)\,T(z,m) + E_d^t(z,m)\,. \tag{7.56}$$

Solving Eqs. (7.45) and (7.46) for $E_u(z)$ and $E_d(m)$ and comparing the result with Eqs. (7.55) and (7.56) gives

$$\begin{bmatrix} T(m,z) & R(m,z) \\ R(z,m) & T(z,m) \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{M}_{--}(m,z) - \mathbf{M}_{-+}(m,z)\,\mathbf{M}_{++}^{-1}(m,z)\,\mathbf{M}_{+-}(m,z) & -\mathbf{M}_{-+}(m,z)\,\mathbf{M}_{++}^{-1}(m,z) \\ \mathbf{M}_{++}^{-1}(m,z)\,\mathbf{M}_{+-}(m,z) & \mathbf{M}_{++}^{-1}(m,z) \end{bmatrix} \tag{7.57}$$

and

$$\left[E_u^t(m,z)\,,\,E_d^t(z,m)\right] =$$

$$\left[E_u^f(m,z) - E_d^f(m,z)\,\mathbf{M}_{++}^{-1}(m,z)\,\mathbf{M}_{+-}(m,z)\,,\; -E_d^f(m,z)\,\mathbf{M}_{++}^{-1}(m,z)\right], \tag{7.58}$$

where for $z = m$

$$\begin{bmatrix} \mathrm{T}(m,m) & R(m,m) \\ R(m,m) & T(m,m) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.59}$$

and

$$\left[E_u^t(m,m)\,,\,E_d^t(m,m)\right] = [0\,,0]\,. \tag{7.60}$$

Equations (7.47) and (7.48) are the *downward global interaction principles* for slab $S[w,z]$, and Eqs. (7.55) and (7.56) are the *upward global interaction principles* for $S[z,m]$. The sense of direction comes from the direction of integration as we "build" a slab starting at either the top or bottom boundaries. Thus in Eqs. (7.47) and (7.48) we think of starting at depth w and integrating Eqs. (7.26) and (7.28) downward to depth z, starting with initial conditions (7.27) and (7.29), in order to obtain the fundamental solution $\underline{\mathbf{M}}(w,z)$. For slab $S[z,m]$ we start at depth m and integrate upward to depth z to obtain $\underline{\mathbf{M}}(m,z)$.

Equations (7.47)-(7.48) and (7.55)-(7.56) are also called the *transport solutions* for their respective slabs $S[w,z]$ and $S[z,m]$. Taken together, these equations constitute the transport solution of the two-flow equations within the water body $S[w,m]$. Note that these equations, viewed together, give us four equations for the two unknown internal irradiances $E_u(z)$ and $E_d(z)$, and for the two unknown response irradiances of the water body, $E_u(w)$ and

$E_d(m)$, in terms of the incident irradiances $E_d(w)$ and $E_u(m)$, and the internal source contributions. Equations (7.51) and (7.57) show how the standard operators can be obtained from the fundamental operators $\underline{\mathbf{M}}(w,z)$ and $\underline{\mathbf{M}}(m,z)$. Equations (7.52) and (7.58) show how the transport internal-source contributions $\boldsymbol{E}^{\mathrm{t}}$ are obtained from the fundamental internal-source contributions $\boldsymbol{E}^{\mathrm{f}}$.

Comments on notation

The reader already may have inferred the rules for interpreting the depth arguments of the various *R*'s, *T*'s, and *E*'s seen in the transport solution. Recall that we always have $a \le w \le z_1 \le z_2 \le m \le b$. In our notation, $R(z_1,z_2)$ stands for the reflectance of slab $S[z_1,z_2]$ for downward irradiance incident at depth z_1, which yields upward irradiance also at level z_1. If we write $R(z_2,z_1)$, we are still working with slab $S[z_1,z_2]$, but $R(z_2,z_1)$ is the reflectance of $S[z_1,z_2]$ for upward irradiance incident at z_2 and leaving the slab in a downward direction at z_2. Likewise $T(z_2,z_1)$ is the transmittance of $S[z_1,z_2]$ for incident upward irradiance at z_2 leaving the slab at z_1,whereas $T(z_1,z_2)$ is for the transmission of downward irradiance from z_1 to z_2.

$E_u^{\mathrm{t}}(z_2,z_1)$ and $E_d^{\mathrm{t}}(z_1,z_2)$ both refer to slab $S[z_1,z_2]$, but $E_u^{\mathrm{t}}(z_2,z_1)$ is the source-induced upward irradiance leaving the slab at level z_1, whereas $E_d^{\mathrm{t}}(z_1,z_2)$ is the source-induced downward irradiance leaving the slab at depth z_2. Strictly speaking, the "u" and "d" subscripts on E^{t} are redundant, since we can deduce the sense of the irradiance from the order of the depth arguments. However, we shall retain the "u" and "d" for notational consistency with E_u and E_d elsewhere in the equations. A little bit of redundancy is sometimes a good thing. We should also remember that $E_u^{\mathrm{f}}(z_1,z_2)$ and $E_d^{\mathrm{f}}(z_1,z_2)$ each have the same order of depth arguments, since these terms represent irradiances at one depth carried to another depth by the fundamental operator.

Generally, then, $R(u,v)$, $T(u,v)$ and $E^{\mathrm{t}}(u,v)$ are associated with slab $S[r,s]$, where r is the lesser of u and v, and s is the greater of u and v, since the notation $S[r,s]$ always has $r \le s$. For $R(u,v)$ irradiance enters and leaves at level u, while for $T(u,v)$, irradiance enters at u and leaves at v. For $E^{\mathrm{t}}(u,v)$, irradiance always leaves at v.

The term "bare slab" in the section title means that we are considering the water body $S[w,m]$ without regard for the effects of its boundaries. In essence, we have mathematically peeled off the air-water surface $S[a,w]$ and the bottom $S[m,b]$ to leave an "unbounded" or "bare" slab of water, $S[w,m]$. Our next task is to learn how to account for the effects of the boundaries, which are always physically present.

7.7 The Transport Solution for Bounded Slabs

The transport solutions of the previous section were obtained by consideration of only the water body $S[w,m]$. We now must learn how to incorporate the effects of the air-water surface and of the bottom boundary on the solution within the water body.

We first consider a composite medium $S[a,z]$ consisting of the air-water surface $S[a,w]$ appended to a slab of water $S[w,z]$, where $w \le z \le m$. We write $S[a,z] = S[a,w] \cup S[w,z]$, where $\cup$ indicates the union of the two slabs. We assume that we know the transport solutions for each of $S[a,w]$ and $S[w,z]$ individually, i.e. we know all of the terms $r(a,w)$, $R(w,z)$, etc., which appear in the transport solutions (7.1)-(7.2), (7.47)-(7.48), and (7.55)-(7.56).

We now ask two questions. First, if we know the *external incident irradiances* $E_d(a)$ and $E_u(z)$, can we find the *internal response irradiances* $E_u(w)$ and $E_d(w)$ at the interface between the slabs? Second, can we find the *external response irradiances* $E_u(a)$ and $E_d(z)$ for the composite slab? The first question will be answered when we develop the *imbed rules* for composite media. Knowing $E_u(w)$ and $E_d(w)$, we can then find the external responses; this development leads to the *union rules* for composite media.

Imbed rules (downward case)

For reference, the most general[1] transport solution for $S[a,w]$ is

$$E_u(a) = E_u(w)\, t(w,a) + E_d(a)\, r(a,w) + E_u^t(w,a) \tag{7.61}$$

$$E_d(w) = E_u(w)\, r(w,a) + E_d(a)\, t(a,w) + E_d^t(a,w) \,. \tag{7.62}$$

For $S[w,z]$ we have

$$E_u(w) = E_u(z)\, T(z,w) + E_d(w)\, R(w,z) + E_u^t(z,w) \tag{7.63}$$

$$E_d(z) = E_u(z)\, R(z,w) + E_d(w)\, T(w,z) + E_d^t(w,z) \,. \tag{7.64}$$

---

[1]We include the possibility of internal sources in $S[a,w]$ so that the results being derived will hold for the union of *any* two slabs, not just the air-water surface plus a layer of water. If $S[a,w]$ represents just a discontinuity in the inherent optical properties, the internal-source terms are zero, and Eqs. (7.61) and (7.62) reduce to the boundary conditions (7.1) and (7.2)

We can now solve Eqs. (7.62) and (7.63) for the internal responses $E_u(w)$ and $E_d(w)$ in terms of the incident irradiances $E_d(a)$ and $E_u(z)$. The results are

$$E_u(w) = E_u(z)\,\mathbf{T}(z,w,a) + E_d(a)\,\mathbf{R}(a,w,z) + \mathbf{E}(z,w,a) \tag{7.65}$$

$$E_d(w) = E_u(z)\,\mathbf{R}(z,w,a) + E_d(a)\,\mathbf{T}(a,w,z) + \mathbf{E}(a,w,z)\,, \tag{7.66}$$

where

$$\mathbf{T}(z,w,a) \equiv T(z,w)\left[1 - r(w,a)\,R(w,z)\right]^{-1} \tag{7.67}$$

$$\mathbf{R}(z,w,a) \equiv \mathbf{T}(z,w,a)\,r(w,a) \tag{7.68}$$

$$\mathbf{E}(z,w,a) \equiv \left[E_u^t(z,w) + E_d^t(a,w)\,R(w,z)\right]\left[1 - r(w,a)\,R(w,z)\right]^{-1} \tag{7.69}$$

and

$$\mathbf{T}(a,w,z) \equiv t(a,w)\left[1 - R(w,z)\,r(w,a)\right]^{-1} \tag{7.70}$$

$$\mathbf{R}(a,w,z) \equiv \mathbf{T}(a,w,z)\,R(w,z) \tag{7.71}$$

$$\mathbf{E}(a,w,z) \equiv \left[E_d^t(a,w) + E_u^t(z,w)\,r(w,a)\right]\left[1 - R(w,z)\,r(w,a)\right]^{-1}. \tag{7.72}$$

Equations (7.65) and (7.66) are the *invariant imbedding relations* for $S[a,z]$, written for a general internal level w, $a \le w \le z$. As a mnemonic aid, note that level w, which appears on the left hand side of the equations, is imbedded within (lies between) depths a and z. The **R**, **T**, and **E** functions are respectively the *complete reflectances*, *complete transmittances* and *complete source-induced irradiances* for $S[a,z]$. Equations (7.67)-(7.72) are the *imbed rules* for the composite medium. *The imbed rules show how the standard operators for two slabs are combined to generate complete operators for the composite slab.*

We have written the reflectances and transmittances for $S[a,w]$ as lower case letters, e.g. $r(a,w)$, as is consistent with our notation for the air-water surface. If we wish to regard $S[a,w]$ as an arbitrary slab of water, we could write the standard reflectances and transmittances as $R(a,w)$, etc., in parallel with our notation for $S[w,z]$.

We also have taken care to maintain the order of the reflectances and transmittances when developing the imbed rules. Note for example, that we write "$r(w,a)R(w,z)$" in Eq. (7.67), but "$R(w,z)r(w,a)$" in Eq. (7.70). We do this in anticipation of using these equations again in the solution of the RTE in Chapter 8. There, the reflectances and transmittances will be matrices, and will not commute. Here, of course, $r(w,a)$ and $R(w,z)$ are numbers and the two products are equal.

Union rules (downward case)

Now that we know the internal irradiances at the interface level w between the two slabs, we can use Eqs. (7.65) and (7.66) to eliminate $E_u(w)$ and $E_d(w)$ in Eqs. (7.61) and (7.64). This yields the external response irradiances $E_u(a)$ and $E_d(z)$ in terms of the incident irradiances. The results are

$$E_u(a) = E_u(z)\,T(z,a) + E_d(a)\,R(a,z) + E_u^t(z,a) \tag{7.73}$$

$$E_d(z) = E_u(z)\,R(z,a) + E_d(a)\,T(a,z) + E_d^t(a,z)\,, \tag{7.74}$$

where

$$T(z,a) = \mathbf{T}(z,w,a)\,t(w,a) \tag{7.75}$$

$$R(a,z) = r(a,w) + \mathbf{R}(a,w,z)\,t(w,a) \tag{7.76}$$

$$E_u^t(z,a) = E_u^t(w,a) + \mathbf{E}(z,w,a)\,t(w,a)\,. \tag{7.77}$$

and

$$T(a,z) = \mathbf{T}(a,w,z)\,T(w,z) \tag{7.78}$$

$$R(z,a) = R(z,w) + \mathbf{R}(z,w,a)\,T(w,z) \tag{7.79}$$

$$E_d^t(a,z) = E_d^t(w,z) + \mathbf{E}(a,w,z)\,T(w,z)\,. \tag{7.80}$$

Equations (7.73) and (7.74) are the *global interaction principles for the composite slab* $S[a,z] = S[a,w] \cup S[w,z]$. These equations show how the entire slab $S[a,z]$ interacts with incident irradiances to produce response irradiances. Equations (7.75)-(7.80) are the associated *union rules. The union rules*, along with the definitions in Eqs. (7.67)-(7.72), *show how the standard operators for two slabs are combined to generate standard operators for the composite slab.* We can imagine holding a and w fixed, and "constructing" the water body by letting z increase downward starting at level w. For this reason, we call the above imbed and union rules those for the "downward case."

Imbed and union rules (upward case)

We can also think of starting at depth m and constructing the water body by letting z move upward while the lower boundary $S[m,b]$ is held fixed. We then obtain complementary formulas for the composite slab $S[z,b] = S[z,m] \cup S[m,b]$. These formulas are derived in exactly the same way as those for the composite slab $S[a,z]$. These "upward case" formulas are presented here for completeness.

The invariant imbedding relations for $S[z,b] = S[z,m] \cup S[m,b]$ are

$$E_u(m) = E_u(b)\,\mathbf{T}(b,m,z) + E_d(z)\,\mathbf{R}(z,m,b) + \mathbf{E}(b,m,z) \tag{7.81}$$

$$E_d(m) = E_u(b)\,\mathbf{R}(b,m,z) + E_d(z)\,\mathbf{T}(z,m,b) + \mathbf{E}(z,m,b)\,. \tag{7.82}$$

The associated *imbed rules* for $S[z,b]$ are

$$\mathbf{T}(b,m,z) \equiv t(b,m)\left[1 - R(m,z)\,r(m,b)\right]^{-1} \tag{7.83}$$

$$\mathbf{R}(b,m,z) \equiv \mathbf{T}(b,m,z)\,R(m,z) \tag{7.84}$$

$$\mathbf{E}(b,m,z) \equiv \left[E_u^t(b,m) + E_d^t(z,m)\,r(m,b)\right]\left[1 - R(m,z)\,r(m,b)\right]^{-1} \tag{7.85}$$

and

$$\mathbf{T}(z,m,b) \equiv T(z,m)\left[1 - r(m,b)\,R(m,z)\right]^{-1} \tag{7.86}$$

$$\mathbf{R}(z,m,b) \equiv \mathbf{T}(z,m,b)\,r(m,b) \tag{7.87}$$

$$\mathbf{E}(z,m,b) \equiv \left[E_d^t(z,m) + E_u^t(b,m)\,R(m,z)\right]\left[1 - r(m,b)\,R(m,z)\right]^{-1}. \tag{7.88}$$

We have written these equations in a general form that allows the bottom boundary $S[m,b]$ to be partially transparent and to have internal sources. For an opaque reflecting bottom such as was considered in Section 7.2, $t(m,b) = t(b,m) = r(b,m) = 0$, $r(m,b) = R$, and $E_d^t(m,b) = E_u^t(b,m) = 0$.

The *global interaction principles* for $S[z,b]$ are

$$E_u(z) = E_u(b)\,T(b,z) + E_d(z)\,R(z,b) + E_u^t(b,z) \tag{7.89}$$

$$E_d(b) = E_u(b)\,R(b,z) + E_d(z)\,T(z,b) + E_d^t(z,b)\,, \tag{7.90}$$

and the associated *union rules* are

$$T(b,z) = \mathbf{T}(b,m,z)\,T(m,z) \tag{7.91}$$

$$R(z,b) = R(z,m) + \mathbf{R}(z,m,b)\,T(m,z) \tag{7.92}$$

$$E_u^t(b,z) = E_u^t(m,z) + \mathbf{E}(b,m,z)\,T(m,z)\,, \tag{7.93}$$

and

$$T(z,b) = \mathbf{T}(z,m,b)\,t(m,b) \tag{7.94}$$

$$R(b,z) = r(b,m) + \mathbf{R}(b,m,z)\,t(m,b) \tag{7.95}$$

$$E_d^t(z,b) = E_d^t(m,b) + \mathbf{E}(z,m,b)\,t(m,b)\,. \tag{7.96}$$

Imbed and union rules for the entire medium $S[a,b]$

We can now express the irradiances $E_u(z)$ and $E_d(z)$ at any depth z within the water body $S[w,m]$ of the composite medium $S[a,b] = S[a,w] \cup S[w,m] \cup S[m,b]$, $a \le w \le z \le m \le b$. All layers may include internal sources, and the entire medium is irradiated from above by $E_d(a)$ and from below by $E_u(b)$. The desired irradiances are found by solving Eqs. (7.74) and (7.89) to obtain the *invariant imbedding relations* $S[a,b]$:

$$E_u(z) = E_u(b)\,\mathbf{T}(b,z,a) + E_d(a)\,\mathbf{R}(a,z,b) + \mathbf{E}(b,z,a) \tag{7.97}$$

$$E_d(z) = E_u(b)\,\mathbf{R}(b,z,a) + E_d(a)\,\mathbf{T}(a,z,b) + \mathbf{E}(a,z,b)\,. \tag{7.98}$$

The *imbed rules* for $S[a,b]$ are

$$\mathbf{T}(b,z,a) \equiv T(b,z)\left[1 - R(z,a)\,R(z,b)\right]^{-1} \tag{7.99}$$

$$\mathbf{R}(b,z,a) \equiv \mathbf{T}(b,z,a)\,R(z,a) \tag{7.100}$$

$$\mathbf{E}(b,z,a) \equiv \left[E_u^t(b,z) + E_d^t(a,z)\,R(z,b)\right]\left[1 - R(z,a)\,R(z,b)\right]^{-1} \tag{7.101}$$

and

$$\mathbf{T}(a,z,b) \equiv T(a,z)\left[1 - R(z,b)\,R(z,a)\right]^{-1} \tag{7.102}$$

$$\mathbf{R}(a,z,b) \equiv \mathbf{T}(a,z,b)\,R(z,b) \tag{7.103}$$

$$\mathbf{E}(a,z,b) \equiv \left[E_d^t(a,z) + E_u^t(b,z)\,R(z,a)\right]\left[1 - R(z,b)\,R(z,a)\right]^{-1}. \tag{7.104}$$

Finally, we obtain the external response irradiances $E_u(a)$ and $E_d(b)$ in terms of the incident irradiances by solving Eqs. (7.73) and (7.90), with $E_u(z)$ and $E_d(z)$ given by Eqs. (7.97) and (7.98). The resulting *global interaction principles* are

$$E_u(a) = E_u(b)\,T(b,a) + E_d(a)\,R(a,b) + E_u^t(b,a) \tag{7.105}$$

$$E_d(b) = E_u(b)\,R(b,a) + E_d(a)\,T(a,b) + E_d^t(a,b)\,. \tag{7.106}$$

The *union rules* for $S[a,b]$ are

$$T(b,a) = \mathbf{T}(b,z,a)\,T(z,a) \tag{7.107}$$

$$R(a,b) = R(a,z) + \mathbf{R}(a,z,b)\,T(z,a) \tag{7.108}$$

$$E_u^t(b,a) = E_u^t(z,a) + \mathbf{E}(b,z,a)\,T(z,a) \tag{7.109}$$

and

$$T(a,b) = \mathbf{T}(a,z,b)\, T(z,b) \tag{7.110}$$

$$R(b,a) = R(b,z) + \mathbf{R}(b,z,a)\, T(z,b) \tag{7.111}$$

$$E_d^t(a,b) = E_d^t(z,b) + \mathbf{E}(a,z,b)\, T(z,b)\,. \tag{7.112}$$

We have now completed the general theoretical framework upon which the numerical procedure of Section 7.2 rests. *These results will play a central role in the solution of the RTE in Chapter 8.* It remains only to develop a self-contained numerical procedure for computing the needed R, T, and $E^{\,t}$ functions for arbitrary water layers and bounding surfaces; such a procedure will generalize Eqs. (7.9) to (7.12).

7.8 Differential Equations for the Standard Operators

We saw in Section 7.5 how to obtain the fundamental solution by integration of Eqs. (7.38) and (7.41), and we saw in Section 7.6 how the standard operators can be obtained from the fundamental operators. However, it is numerically advantageous to develop a means for directly computing the standard operators, without first computing the fundamental operators. The reason in part is because the fundamental operator $\underline{\mathbf{M}}(z_o,z)$ behaves approximately exponentially with depth z, as will be seen in Chapter 9. Such behavior can cause numerical difficulties if calculations are carried to great depths. The R and T standard operators, on the other hand, are bounded by 0 and 1 since they represent physical reflectances and transmittances. This benign depth behavior leads to numerically well behaved algorithms for the computation of the standard operators. In addition, direct computation of the standard operators insures the independence of the transport and fundamental solution methods, so that one may serve as a check on the other.

We now show how to determine the eight standard transmittance and reflectance operators, and the four transport source-induced irradiances needed in the interaction statements (7.73)-(7.74) and (7.89)-(7.90).

These standard operators are governed by twelve differential equations, which group naturally into two sextets of equations – an upward family and a downward family. Each sextet of equations splits naturally into a major trio and a minor trio. We shall derive the major downward trio of equations; the others follow in a similar fashion.

Consider the slab $S[a,z] = S[a,w] \cup S[w,z]$. The four-step derivation of the major downward trio for $S[a,z]$ begins by differentiating the interaction principle (7.74) with respect to z to obtain

$$\frac{d}{dz} E_d(z) = E_u(z) \frac{d}{dz} R(z,a) + \left[\frac{d}{dz} E_u(z)\right] R(z,a)$$
$$+ E_d(a) \frac{d}{dz} T(a,z) + \frac{d}{dz} E_d^t(a,z) .$$

Second, use the two-flow equations (7.3) and (7.4) to replace the derivatives of $E_u(z)$ and $E_d(z)$. The result is

$$E_d(z)\, \tau_{dd}(z) + E_u(z)\, \rho_{ud}(z) + E_{od}^S(z)$$
$$= E_u(z) \frac{d}{dz} R(z,a) + \left[-E_u(z)\, \tau_{uu}(z) - E_d(z)\, \rho_{du}(z) - E_{ou}^S(z)\right] R(z,a)$$
$$+ E_d(a) \frac{d}{dz} T(a,z) + \frac{d}{dz} E_d^t(a,z) .$$

Third, use Eq. (7.74) to replace the occurrences of the response irradiance $E_d(z)$ in the previous equation and group the terms to get

$$\begin{aligned} 0 = {} & E_u(z) \left\{\frac{d}{dz} R(z,a) - R(z,a)\left[\tau_{dd}(z) + \rho_{du}(z)\, R(z,a)\right] - \rho_{ud}(z) - \tau_{uu}(z)\, R(z,a)\right\} \\ & + E_d(a) \left\{\frac{d}{dz} T(a,z) - T(a,z)\left[\tau_{dd}(z) - \rho_{du}(z)\, R(z,a)\right]\right\} \\ & + \left\{\frac{d}{dz} E_d^t(a,z) - E_d^t(a,z)\left[\tau_{dd}(z) + \rho_{du}(z)\, R(z,a)\right] - E_{od}^S(z) - E_{ou}^S(z)\, R(z,a)\right\} . \end{aligned} \tag{7.113}$$

Fourth, recognize that the incident irradiance $E_d(a)$ is arbitrary. Likewise, note that the internal sources $E_{ou}{}^S(z)$ and $E_{od}{}^S(z)$, which generate $E_d{}^t(a,z)$, can be chosen arbitrarily and independently of $E_d(a)$. In particular, Eq. (7.113) still holds if $E_d(a)$ is zero, and $E_{ou}{}^S$ and $E_{od}{}^S$ are nonzero. Similarly, Eq. (7.113) still holds if $E_d(a) \neq 0$ and $E_{ou}{}^S = E_{od}{}^S = 0$, so that $E_d{}^t(a,z) = 0$ throughout $S[a,z]$. Hence, each of the three groups of terms in Eq. (7.113) must individually be zero. [If $a + b + c = 0$, $a + b = 0$, and $a + c = 0$, then it follows that $a = b = c = 0$.] Moreover, since $E_u(z)$ is in general nonzero [owing to any or all of $E_d(a) \neq 0$, $E_{ou}{}^S \neq 0$, or $E_{od}{}^S \neq 0$ being true], it follows that the quantities enclosed in each pair of braces in Eq. (7.113) are identically zero for all z in $w \leq z \leq m$. We thus arrive at the very important *Riccati differential equations* for $R(z,a)$, $T(a,z)$, and $E_d{}^t(a,z)$:

$$\frac{d}{dz} R(z,a) = R(z,a)\left[\tau_{dd}(z) + \rho_{du}(z)\, R(z,a)\right] + \rho_{ud}(z) + \tau_{uu}(z)\, R(z,a) \tag{7.114}$$

$$\frac{d}{dz} T(a,z) = T(a,z)\left[\tau_{dd}(z) + \rho_{du}(z)\, R(z,a)\right] \tag{7.115}$$

$$\frac{d}{dz} E_d^t(a,z) = E_d^t(a,z)\left[\tau_{dd}(z) + \rho_{du}(z)\, R(z,a)\right] + E_{od}^S(z) + E_{ou}^S(z)\, R(z,a)\,. \tag{7.116}$$

These equations are the *major downward trio.*

By similar arguments, beginning with a differentiation of Eq. (7.73), we arrive at the *minor downward trio*:

$$\frac{d}{dz} T(z,a) = \left[\tau_{uu}(z) + R(z,a)\, \rho_{du}(z)\right] T(z,a) \tag{7.117}$$

$$\frac{d}{dz} R(a,z) = T(a,z)\, \rho_{du}(z)\, T(z,a) \tag{7.118}$$

$$\frac{d}{dz} E_u^t(z,a) = \left[E_{ou}^S(z) + E_d^t(a,z)\, \rho_{du}(z)\right] T(z,a)\,. \tag{7.119}$$

First order, quadratically nonlinear differential equations like Eq. (7.114) are called Riccati equations. We shall for convenience refer to the entire set of equations as "the Riccati equations," because of the leading role played by Eq. (7.114), and by Eq. (7.121) below.

The *downward sextet* (7.114)-(7.119) derives its name from the observation that these differential equations can be integrated in a downward sweep beginning at $z = w$ and ending at $z = m$. The required initial values are

$$\begin{aligned}
R(w,a) &= r(w,a) & [= 0]\\
R(a,w) &= r(a,w) & [= 0]\\
T(a,w) &= t(a,w) & [= 1]\\
T(w,a) &= t(w,a) & [= 1]\\
E_d^t(a,w) &= E_d^t(a,w)_o & [= 0]\\
E_u^t(w,a) &= E_u^t(w,a)_o & [= 0]\,.
\end{aligned} \tag{7.120}$$

If $S[a,w]$ represents an air-water surface, the r and t values are computed as in Chapter 4. The initial values $E_d^t(a,w)_o$ and $E_u^t(w,a)_o$, which describe the effects of any internal sources within the surface layer $S[a,w]$, must be determined from independent calculations. For an air-water surface, these quantities are zero. The values in square brackets show the initial conditions corresponding to a *transparent boundary* with no internal sources, i.e. to the case of a bare slab. These parenthetical values are included here to facilitate the later comparison of the present equations with their radiance counterparts in Section 8.7.

Note that Eq. (7.114) for $R(z,a)$ is autonomous, but that the other two equations of the major trio depend on $R(z,a)$. The equations of the minor trio depend on the three quantities of the major trio, but not vice versa.

If we begin with a differentiation of Eq. (7.89), we obtain the *major upward trio:*

$$-\frac{d}{dz}R(z,b) = R(z,b)\left[\tau_{uu}(z) + \rho_{ud}(z)\,R(z,b)\right] + \rho_{du}(z) + \tau_{dd}(z)\,R(z,b) \tag{7.121}$$

$$-\frac{d}{dz}T(b,z) = T(b,z)\left[\tau_{uu}(z) + \rho_{ud}(z)\,R(z,b)\right] \tag{7.122}$$

$$-\frac{d}{dz}E_u^t(b,z) = E_u^t(b,z)\left[\tau_{uu}(z) + \rho_{ud}(z)\,R(z,b)\right] + E_{ou}^S(z) + E_{od}^S(z)\,R(z,b). \tag{7.123}$$

The *minor upward trio* arising from Eq. (7.90) is

$$-\frac{d}{dz}T(z,b) = \left[\tau_{dd}(z) + R(z,b)\,\rho_{ud}(z)\right]T(z,b) \tag{7.124}$$

$$-\frac{d}{dz}R(b,z) = T(b,z)\,\rho_{ud}(z)\,T(z,b) \tag{7.125}$$

$$-\frac{d}{dz}E_d^t(z,b) = \left[E_{od}^S(z) + E_u^t(b,z)\,\rho_{ud}(z)\right]T(z,b)\,. \tag{7.126}$$

Now we start the integration at level m and integrate upward to level w. The associated initial values are

$$\begin{aligned} R(m,b) &= r(m,b) && [=0] \\ R(b,m) &= r(b,m) && [=0] \\ T(b,m) &= t(b,m) && [=1] \\ T(m,b) &= t(m,b) && [=1] \\ E_u^t(b,m) &= E_u^t(b,m)_o && [=0] \\ E_d^t(m,b) &= E_d^t(m,b)_o && [=0]\,. \end{aligned} \tag{7.127}$$

The r's, t's, $E_u^t(b,m)_o$ and $E_d^t(m,b)_o$ appropriate to the lower boundary $S[m,b]$ must be supplied. For the opaque Lambertian bottom of Section 7.2, we had $R(m,b) = R$, and all other boundary quantities equal to zero [recall Eqs.

(7.11) and (7.12)]. As in Eq. (7.120), the values in brackets in Eq. (7.127) give the initial conditions appropriate for a transparent bottom.

Summary of the transport solution

We now have in hand a solution algorithm for the most general formulation of the two-flow equations, including arbitrary upper and lower boundaries, and the presence of internal sources even within the boundaries themselves. It is worthwhile to summarize the computation involved in the transport solution procedure:

(i) Integrate the downward sextet of Riccati differential equations (7.114)-(7.119) in a downward sweep from depth w to depth m, starting with the initial values shown in Eq. (7.120). Likewise, integrate the upward sextet (7.121)-(7.126) from depth m to depth w, starting with the initial conditions of Eq. (7.127). Save the results of these integrations at depths $w \equiv z_1, z_2, ..., z_k \equiv m$, where $z_1, ..., z_k$ are those preselected depths within the water body at which the solution irradiances are desired.

(ii) Evaluate the six complete transmittance, reflectance, and source-induced irradiance operators seen in the imbed rules (7.99)-(7.104). These operators are evaluated at all depths $z_1, ..., z_k$ where a solution is desired. That is, we compute and save $\mathbf{T}(b,w,a)$, $\mathbf{T}(b,z_2,a)$, ..., $\mathbf{T}(b,m,a)$, and so on. Note that these computations make use of the standard operators governed by the major upward and downward trios of Riccati equations.

(iii) Compute the upward and downward irradiances at depths $z_1, ..., z_k$ using the invariant imbedding relations (7.97) and (7.98).

(iv) Compute the external responses $E_u(a)$ and $E_d(b)$ using the global interaction principles (7.105) and (7.106), along with the union rules (7.107)-(7.112) evaluated at, say, $z_1 = w$. Note that these union rules use the standard operators computed from the minor upward and downward trios of Riccati equations.

The complete solution of the two-flow problem has now been achieved.

7.9 Summary

We have now penetrated deeply into the mathematical structure of radiative transfer theory, although not nearly so deeply as can be done. We could continue to investigate the conceptual constructions of this chapter – invariant imbedding relations, imbed and union rules, standard and complete operators, fundamental and transport solutions, and so on. However, to do so would take us beyond what is needed to reach our primary goal of solving the RTE. The above study is continued at the irradiance level in Preisendorfer (1987) and in Preisendorfer and Mobley (1988).

Our mathematical efforts already have been rewarded in the insights we have obtained. For example, we have seen that the complete operators *algebraically* incorporate all orders of multiple scattering into the solution irradiances. We have discovered different approaches – the transport and fundamental solutions – to the same problem, and we have established relationships between these approaches. Moreover, our mathematical developments have led to a specific algorithm for solving the most general two-flow problem. Indeed, this algorithm is so general and powerful that the RTE itself will yield to it in Chapter 8. Obtaining numerical solutions of the RTE will be our greatest reward for the work expended in this chapter.

However, in spite of the detailed developments above, we have neglected one small facet of the theory – the name itself. What does *invariant imbedding* mean: in particular, what is invariant, and what is imbedded in what? The name itself is due to Bellman and Kalaba (1956), and the analytic form of the invariant imbedding relation first appeared in Preisendorfer (1958b). The underlying idea can be traced to Ambarzumian's (1943) problem and his revolutionary insight into its solution. He was interested in computing the reflectance of an infinitely deep stellar atmosphere. The "classical" approach to this problem would be to solve the RTE itself within the atmosphere (using, for example, Monte Carlo methods) given the incident radiance, and thereby eventually obtain the radiance leaving the atmosphere, from which the reflectance then could be computed. Ambarzumian's insight was this: the reflectance of an infinitely deep stellar atmosphere will be unchanged (that is to say, will be invariant) if another layer is added to or subtracted from the atmosphere. He thus "imbedded" his problem (the reflectance of an atmosphere of depth z) in a related problem (the reflectance of an atmosphere of depth $z + dz$). He was then able to derive a functional equation relating the reflectance of an atmosphere of depth z to the reflectance of an atmosphere of depth $z + dz$, in the limit

$z \rightarrow \infty$. The equation governing the reflectance was then solved directly, without the necessity of solving the RTE itself.

Subsequent development of Ambarzumian's method showed that the *functional forms of the equations* governing the reflectances (and the transmittances, etc.) *remain invariant* even for inhomogeneous slabs of finite thickness. Clearly, if we change the depth of a finite layer of water, the magnitude of the reflectance will change. Therefore, the modern use of "invariance" refers to the forms of the governing equations, and not to the numerical values of the computed quantities. Preisendorfer (1965, Section 49) expands further on these generalizations.

Ambarzumian's method is clearly an ancestor of what we have done. Recall the solution procedure of Section 7.8. There we developed differential equations governing not the irradiances themselves, but rather certain reflectances, transmittances, and internal sources (the standard operators) of the irradiances. Our computational effort is expended in solving these differential equations, which describe how the various standard operators build up as we add infinitesimal layers of water to boundary layers of known properties. We eventually obtained the irradiances within the water body from the standard operators (via the imbed rules for the complete operators and the invariant imbedding relations), and not from an explicit integration of the two-flow equations themselves.

Note from Eq. (7.89) that $R(z,b)$ is just the irradiance reflectance $E_u(z)/E_d(z)$ of a source-free medium $S[z,b]$ that has no irradiance incident from below. If all that we desire is this irradiance reflectance, then we need integrate only Eq. (7.121). It is not necessary to explicitly compute the irradiances in order to obtain $R(z,b)$. Letting $m = b \rightarrow \infty$ gives Ambarzumian's problem. Similarly, we can obtain the reflectance $R(a,b)$ of an entire, arbitrary medium $S[a,b]$ from Eq. (7.108) after integrating the Riccati Eqs. (7.114)-(7.127). We need not compute $E_u(a)$ itself.

Invariant imbedding theory has undergone much refinement since Ambarzumian's seminal idea. The theory has found wide application in *linear transport problems*, including radiative transfer, neutron diffusion, water wave propagation, electrical engineering, and acoustics. An elementary example of the use of invariant imbedding is given in Chapter 1 of Bellman, *et al.* (1963). The review paper by Bellman, *et al.* (1960) is excellent. Chandrasekhar (1960) discusses principles of invariance in considerable detail. A rigorous mathematical development of the theory in the setting of hydrologic optics is found in the works of Preisendorfer, especially *H.O.* volumes II and IV.

7.10 Problems

7.1. Draw a figure similar to Fig. 7.1 to provide a graphical interpretation of the complete operator $\mathbf{E}(a,z,b)$ defined in Eq. (7.104).

7.2. Draw a figure similar to Fig. 7.1 to provide a graphical interpretation of the standard operator $R(a,b)$ defined in Eq. (7.108). Be sure to "graphically expand" the complete operators contained within $R(a,b)$.

7.3. Explain in words the physical meaning of the relationship between $E_d^t(w,z)$ and $E_d^f(w,z)$ as seen in Eq. (7.52).

7.4. Carry out the steps of the derivation of the minor downward trio of Riccati Eqs. (7.117)-(7.119).

7.5. Consider the union $S[w,m] = S[w,z] \cup S[z,m]$ of two optically thin slabs of water. By "optically thin" we mean that the reflectances are near zero and the transmittances are near one, i.e. $R(w,z) \approx 0$, $T(w,z) \approx 1$, and so on. Under these conditions, show that

$$R(w,m) \approx R(w,z) + R(z,m)$$

and

$$T(w,m) \approx T(w,z)\, T(z,m)\,.$$

In other words, show that *for optically thin slabs, reflectances add and transmittances multiply.*

7.6. Consider an optically thin body of water $S[a,z]$. Then, for example, Eq. (7.118) reduces to

$$\frac{d}{dz} R(a,z) \approx \rho_{du}(z)\,.$$

This equation reinforces our interpretation of ρ_{du} as the downward-to-upward irradiance reflectance, per meter of water, of an infinitesimally thin layer of water. Examine the other Riccati equations in this manner, to see if they yield similar physical interpretations.

7.7. Let $S[z,b]$ be an optically very thick cloud layer. In our coordinate system with z positive downward, the cloud layer will increase in thickness if b increases or if z decreases. Now reason, as Ambarzumian did, that after the cloud layer becomes very thick, any further increase in thickness will not change the reflectance of the cloud layer as seen from above, $R(z,b)$. Use this insight to derive an explicit formula for $R(z,b)$ in terms of the four τ's and ρ's. Now, for convenience, think of the same cloud layer as being $S[a,z]$. Derive a formula for $R(z,a)$, the irradiance reflectance of the thick cloud layer as seen from below. Is $R(z,b) = R(a,z)$? In other words, does the same cloud layer have the same irradiance reflectance when viewed from above or from below? Discuss.

Next Chapter >>