This page develops the mathematical tools needed to specify directions and angles in three-dimensional space. These mathematical concepts are fundamental to the specification of how much light there is and what direction it is traveling.

Directions

We will have frequent need to specify directions. In order to do this in Euclidean three-dimensional space, let $\hat{i}_1, \hat{i}_2$, and $\hat{i}_3$ be three mutually perpendicular unit vectors that define a right-handed Cartesian coordinate system. We choose $\hat{i}_1$ to be in the direction that the wind is blowing over the ocean surface, (i.e. $\hat{i}_1$ points downwind), and $\hat{i}_3$ points downward into the water column, perpendicular to the mean position of the water surface; $\hat{i}_2$ is then in the direction given by the cross (or vector) product $\hat{i}_2 = \hat{i}_3 \times \hat{i}_1$. The choice of a "wind-based" coordinate system simplifies the mathematical specification of sea-surface wave spectra via along-wind and cross-wind statistics. How this is done will be seen in the Surfaces chapter, beginning with the page on Cox-Munk sea-surface slope statistics. The choice of $\hat{i}_3$ pointing downward is natural in oceanography, where depths are customarily measured as positive downward from an origin at mean sea level.

With the choice of $\hat{i}_1, \hat{i}_2$, and $\hat{i}_3$, an arbitrary direction can be specified as follows. Let $\hat{\xi}$ denote a unit vector pointing in the desired direction. The vector $\hat{\xi}$ has components ξ_1, ξ_2 and ξ_3 in the $\hat{i}_1, \hat{i}_2$, and $\hat{i}_3$ directions, respectively. We can therefore write $\hat{\xi} = \xi_1 \hat{i}_1 + \xi_2 \hat{i}_2 + \xi_3 \hat{i}_3$, or just $\hat{\xi} = (\xi_1, \xi_2, \xi_3)$ for notational convenience. Note that because $\hat{\xi}$ is of unit length, its components satisfy $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$.

An alternative description of $\hat{\xi}$ is given by the polar coordinates θ and ϕ, defined as shown in Fig. **??**. The *nadir angle* θ is measured from the nadir direction $\hat{i}_3$, and the *azimuthal angle* ϕ is measured positive counterclockwise from $\hat{i}_1$, when looking toward the origin along $\hat{i}_3$ (i.e. when looking in the $-\hat{i}_3$ direction). The connection between $\hat{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\hat{\xi} = (\theta, \phi)$ is obtained by inspection of Fig. **??**:

$$\xi_1 = \tag{1}$$

$$\sin\theta \cos\phi \;\, \xi_2 =$$

$$\sin\theta \sin\phi \;\, \xi_3 = \tag{2}$$

$$\cos\theta$$

where θ and ϕ lie in the ranges $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The inverse transformation is

$$\theta = \tag{3}$$

$$\cos^{-1}(\xi_3) \;\, \phi =$$

$$\tan^{-1}\left(\frac{\xi_2}{\xi_1}\right) \tag{4}$$

The polar coordinate form of $\hat{\xi}$ could be written as $\hat{\xi} = (r, \theta, \phi)$, but since the length of r is 1, we drop the radial coordinate for brevity.

Another useful description of $\hat{\xi}$ is obtained using the cosine parameter

$$\mu \equiv \cos\theta = \xi_3. \tag{5}$$

The components of $\hat{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\hat{\xi} = (\mu, \phi)$ are related by

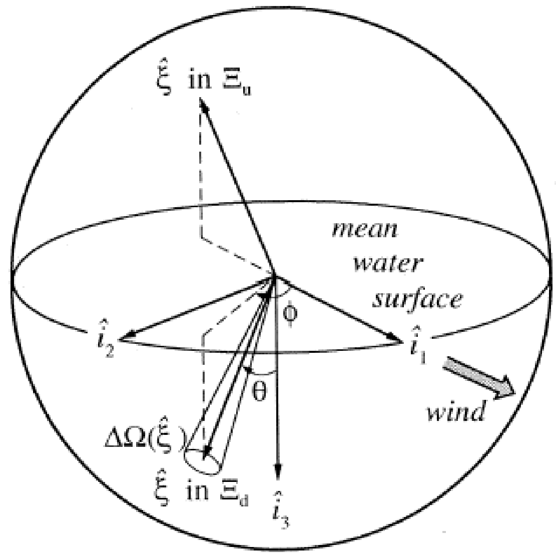

Figure 1: Definition of the polar coordinates (θ, ϕ) and of the upward (Ξ_u) and downward (Ξ_d) hemispheres of directions. $\Delta\Omega(\hat{\xi})$ is an element of solid angle centered on $\hat{\xi}$.

$$\xi_1 = \tag{6}$$

$$(1-\mu^2)^{1/2} \cos\phi \; \xi_2 =$$

$$(1-\mu^2)^{1/2} \sin\phi \; \xi_3 = \tag{7}$$

$$\mu,$$

with μ and ϕ in the ranges $-1 \le \mu \le 1$ and $0 \le \phi < 2\pi$. Hence a direction $\hat{\xi}$ can be represented in three equivalent ways: as (ξ_1, ξ_2, ξ_3) in Cartesian coordinates, and as (θ, ϕ) or (μ, ϕ) in polar coordinates.

The scalar (or dot) product between two direction vectors $\hat{\xi}'$ and $\hat{\xi}$ can be written as

$$\hat{\xi}' \cdot \hat{\xi} = |\hat{\xi}'|\,|\hat{\xi}| \cos\psi = \cos\psi,$$

where ψ is the angle between directions $\hat{\xi}'$ and $\hat{\xi}$, and $|\hat{\xi}|$ denotes the (unit) length of vector $\hat{\xi}$. The scalar product expressed in Cartesian-component form is

$$\hat{\xi}' \cdot \hat{\xi} = \xi'_1\xi_1 + \xi'_2\xi_2 + \xi'_3\xi_3.$$

Equating these representations of $\hat{\xi}' \cdot \hat{\xi}$ and recalling Eqs. (**??**) and (**??**) leads to

$$\cos\psi = \tag{8}$$

$$\xi'_1\xi_1 + \xi'_2\xi_2 + \xi'_3\xi_3 \; =$$

$$\cos\theta' \cos\theta \; + \sin\theta' \sin\theta \, \cos(\phi' - \phi) \; = \tag{9}$$

$$\mu'\mu + \sqrt{1-\mu'^2}\,\sqrt{1-\mu^2}\, \cos(\phi' - \phi)$$

Equation (**??**) gives very useful connections between the various coordinate representations of $\hat{\xi}'$ and $\hat{\xi}$, and the included angle ψ. In particular, this equation allows us to compute the scattering angle ψ when light is scattered from an incident to a final direction.

The set of all directions $\hat{\xi}$ is called the *unit sphere of directions*, which is denoted by Ξ. Referring to polar coordinates, Ξ therefore represents all (θ, ϕ) values such that $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Two subsets of Ξ frequently employed in optical oceanography are the *downward* (subscript d) and *upward* (subscript u) *hemispheres* of directions, Ξ_d and Ξ_u, defined by

$$\begin{aligned} &\Xi_d \\ &\equiv \text{ all } (\theta, \phi) \text{ such that } 0 \leq \theta \leq \pi/2 \text{ and } 0 \leq \phi < 2\pi,\ \Xi_u \\ &\equiv \text{ all } (\theta, \phi) \text{ such that } \pi/2 < \theta \leq \pi \text{ and } 0 \leq \phi < 2\pi. \end{aligned} \tag{10}$$

Solid Angle

Closely related to the specification of directions in three-dimensional space is the concept of *solid angle*, which is an extension of two-dimensional angle measurement. As illustrated in panel (a) of Fig. **??**, the plane angle θ between two radii of a circle of radius r is

$$\theta \equiv \frac{\text{arc length}}{\text{radius}} = \frac{\ell}{r} \qquad \text{(rad)}.$$

The angular measure of a full circle is therefore 2π rad. In panel (b) of Fig. **??**, a patch of area A is shown on the surface of a sphere of radius r. The boundary of A is traced out by a set of directions $\hat{\xi}$. The solid angle Ω of the set of directions defining the patch A is by definition

$$\Omega \equiv \frac{\text{area}}{\text{radius squared}} = \frac{A}{r^2} \qquad \text{(sr)}.$$

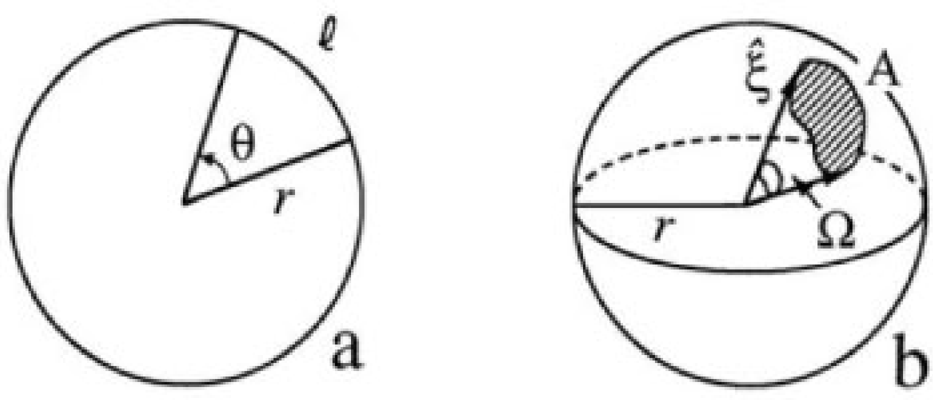

Figure 2: Geometry associated with the definition of plane angle (panel a) and solid angle (panel b).

Since the area of a sphere is $4\pi r^2$, the solid angle measure of the set of all directions is $\Omega(\Xi) = 4\pi$ sr. Note that both plane angle and solid angle are independent of the radii of the respective circle and sphere. Both plane and solid angle are dimensionless numbers. However, they are given "units" of radians and steradians, respectively, to remind us that they are measures of angle.

Consider a simple application of the definition of solid angle and the observation that a full sphere has $4\pi\,\mathrm{sr}$. The area of Brazil is $8.5 \cdot 10^6\,\mathrm{km}^2$ and the area of the earth's surface is $5.1 \cdot 10^8\,\mathrm{km}^2$. The solid angle subtended by Brazil as seen from the center of the earth is then $4\pi\, 8.5 \cdot 10^6 / 5.1 \cdot 10^8 = 0.21\,\mathrm{sr}$.

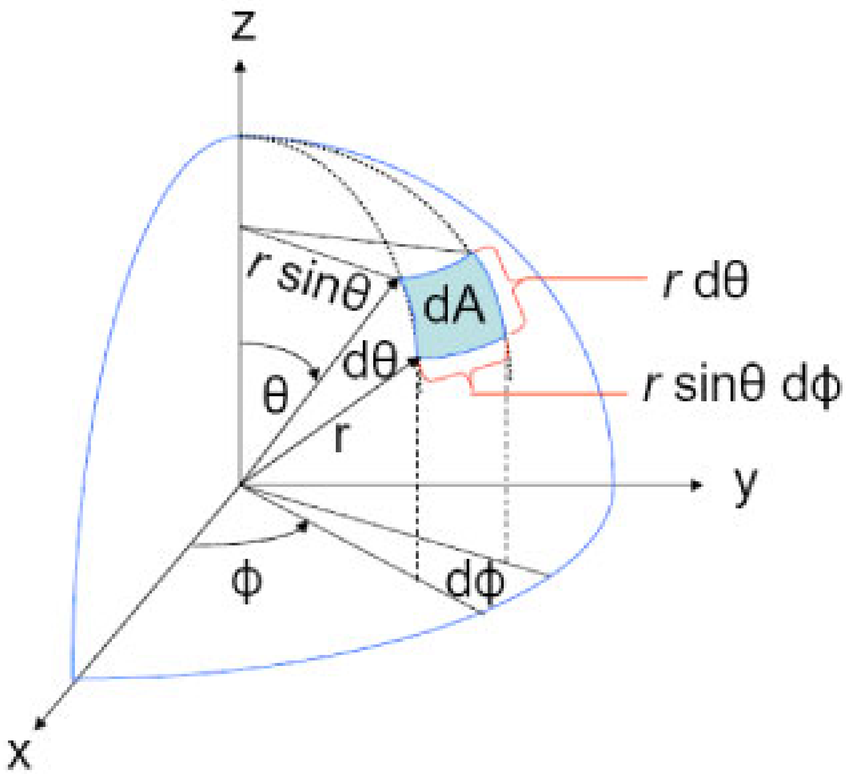

Figure 3: Geometry used to obtain an element of solid angle in spherical coordinates.

The definition of solid angle as area on the surface of a sphere divided by radius of the sphere squared gives us a convenient form for a *differential element of solid angle,* as needed for computations. The blue patch shown in Fig. **??** represents a differential element of area dA on the surface of a sphere of radius r. Simple trigonometry shows that this area is $dA = (r \sin\theta\, d\phi)(r\, d\theta)$. Thus the element of solid angle $d\Omega(\hat{\xi})$ about the direction $\hat{\xi} = (\theta, \phi)$ is given in polar coordinate form by

$$d\Omega(\hat{\xi}) = \tag{11}$$

$$\frac{dA}{r^2} = \frac{(r\,\sin\theta\, d\phi)(r\, d\theta)}{r^2} = \sin\theta\, d\theta\, d\phi \ =$$

$$d\mu\, d\phi \qquad (\mathrm{sr}). \tag{12}$$

(The last equation is correct even though $d\mu = d\cos\theta = -\sin\theta\, d\theta$. When the differential element is used in an integral and variables are changed from (θ, ϕ) to (μ, ϕ), the Jacobian of the transformation involves an absolute value.)

Example: Solid angle of a spherical cap

To illustrate the use of Eq. (**??**), let us compute the solid angle of a "polar cap" of half angle θ, i.e. all (θ', ϕ') such that $0 \leq \theta' \leq \theta$ and $0 \leq \phi' < 2\pi$. Integrating the element of solid angle over this range of (θ', ϕ') gives

$$\Omega_{\text{cap}} = \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\theta} \sin\theta' \, d\theta' \, d\phi' = 2\pi(1-\cos\theta), \tag{13}$$

or

$$\Omega_{\text{cap}} = \int_{\phi'=0}^{2\pi} \int_{\mu'=\mu}^{1} d\mu' \, d\phi' = 2\pi(1-\mu). \tag{14}$$

Note that Ξ_d and Ξ_u are special cases of a spherical cap (having $\theta = \pi/2$), and that $\Omega(\Xi_d) = \Omega(\Xi_u)$ $= 2\pi$ sr.

Dirac Delta functions

It is sometimes convenient to specify directions using the *Dirac delta function*, $\delta(\hat{\xi} - \hat{\xi}_o)$. This peculiar mathematical construction is defined (for our purposes) by

$$\delta(\hat{\xi} - \hat{\xi}_o) \equiv 0 \quad \text{if } \hat{\xi} \neq \hat{\xi}_o, \tag{15}$$

and

$$\int_{\Xi} f(\hat{\xi}) \delta(\hat{\xi} - \hat{\xi}_o) d\Omega(\hat{\xi}) \equiv f(\hat{\xi}_o). \tag{16}$$

Here $f(\hat{\xi})$ is any function of direction. Note that $\delta(\hat{\xi} - \hat{\xi}_o)$ simply "picks out" the particular direction $\hat{\xi}_o$ from all directions in Ξ. Note also in Eq. (**??**) that because the element of solid $d\Omega(\hat{\xi})$ has units of steradians, it follows that $\delta(\hat{\xi} - \hat{\xi}_o)$ has units of inverse steradians.

Equations (**??**) and (**??**) are a symbolic definition of δ. The *mathematical representation* of $\delta(\hat{\xi} - \hat{\xi}_o)$ in spherical coordinates (θ,ϕ) is

$$\delta(\hat{\xi} - \hat{\xi}_{\text{o}}) = \frac{\delta(\theta - \theta_{\text{o}})\delta(\phi - \phi_{\text{o}})}{\sin\theta} \qquad (\text{sr}^{-1}), \tag{17}$$

where $\hat{\xi} = (\theta, \phi), \hat{\xi}_o = (\theta_o, \phi_o)$, and

$$\int_0^{\pi} f(\theta)\delta(\theta - \theta_{\text{o}}) d\theta \equiv$$

$$f(\theta_{\text{o}}) \quad \int_0^{2\pi} f(\phi)\delta(\phi - \phi_{\text{o}}) d\phi \equiv$$

$$f(\phi_{\text{o}}).$$

Note that the $\sin\theta$ in the denominator of Eq. (**??**) is necessary to cancel the $\sin\theta$ factor in the element of solid angle when integrating in polar coordinates. Thus

$$\begin{aligned}
&\int_{\Xi} f(\hat{\xi})\delta(\hat{\xi} - \hat{\xi}_{\text{o}}) d\Omega(\xi) = \\
&\int_0^{2\pi} \int_0^{\pi} f(\theta,\phi) \frac{\delta(\theta - \theta_{\text{o}})\delta(\phi - \phi_{\text{o}})}{\sin\theta} \sin\theta d\theta d\phi = \\
&f(\theta_{\text{o}}, \phi_{\text{o}}) \;=\; f(\hat{\xi}_{\text{o}}).
\end{aligned}$$

Likewise, we can write

$$\delta(\hat{\xi} - \hat{\xi}_{\rm o}) = \delta(\mu - \mu_{\rm o})\delta(\phi - \phi_{\rm o}) \qquad (\mathrm{sr}^{-1}), \tag{18}$$

where

$$\int_{-1}^{1} f(\mu)\delta(\mu - \mu_{\rm o})d\mu \equiv f(\mu_{\rm o}).$$