The 1D, time-independent RTEs are *linear two-point boundary value problems*. That is to say, there are boundary conditions describing the radiance at the top and bottom of the atmosphere or ocean (the two spatial points), and the propagation of radiance within the medium—between the boundaries—is governed by the linear integro-differential RTE. In the oceanographic setting, the upper boundary condition specifies the sky radiance incident onto the sea surface. The lower boundary condition specifies how the sea bottom reflects the downwelling radiance. (The "sea bottom" can be a physical bottom, or the deepest depth in the water column at which the RTE needs to be solved to obtain the radiances above that depth.) In the vector-level equations, the sky radiance specification is in terms of Stokes vectors; for the scalar RTE the sky input is specified by the unpolarized sky radiance.

Two point boundary value problems in general do not have solutions. Consider a simple example:

$$\frac{dy}{dx} = x^2$$

with the two-point boundary conditions

$$y = 0 \text{ at } x = 0 \text{ and } y = 2 \text{ at } x = 1\,.$$

What is the function $y(x)$ that satisfies the differential equation and the two boundary conditions? Integration gives $y = x^3/3$, which satisfies the first boundary condition but not the second. Thus the problem as stated has no solution; i.e., there is no such function $y(x)$. This equation however does have a solution for the two boundary conditions

$$y = 0 \text{ at } x = 0 \text{ and } y = \frac{1}{3} \text{ at } x = 1\,.$$

In the case of the radiative transfer equation and boundary conditions on the radiance at the sea surface and sea bottom (when properly formulated), the equation does have a solution. We know a solution must exist simply because light propagates in the ocean! However, finding that solution is quite difficult. Even the scalar radiative transfer equation (SRTE; Eq. 3 of the The Scalar Radiative Transfer Equation page) considered here is quite difficult to solve.

Exact Analytical Solutions

Exact analytical (i.e., pencil and paper) solutions of the SRTE can be obtained only for very simple situations, such as no scattering. There is no function (that anyone has ever found) that gives

$$L(z, \theta, \phi, \lambda) = f(a, VSF, \text{ sun angle, bottom reflectance, etc.}),$$

where f is some function in which you can "plug in" the absorption coefficient, VSF, and other parameters and get back the radiance. This is true even for very simple situations such as homogeneous water with isotropic scattering. Even the conceptually simple geometry of an isotropically emitting point light source in an infinite homogeneous ocean is unsolved. (A very complicated solution for the the scalar irradiance $E_o(r)$ around an isotropically emitting point source with isotropic scattering does exist; see Davison and Sykes (1957), eq. 5.25 or Mobley (1996), Section 9.2.) This may seem surprising because other point-source problems, such as the electric filed around a point charge or the gravitational field around a point mass, are often very simple. The difference with optics lies in the complications caused by scattering within the medium surrounding the point source (which do not exist for problems like the gravitational field around a point mass).

Beer's Law

If there is no scattering, the SRTE reduces to just

$$\frac{dL(r, \theta, \phi, \lambda)}{dr} = -a(r, \lambda)L(r, \theta, \phi, \lambda) + S(r, \theta, \phi, \lambda) \ . \tag{1}$$

This is a linear, first order, ordinary differential equation, which is easily solved (see any text on differential equations, e.g., Rainville (1964), page 36). If the medium is homogeneous, so that the absorption coefficient and source function do not depend on distance r, the solution of Eq. (1) is (dropping the wavelength and direction arguments for brevity)

$$L(r) = L(0)e^{-ar} + \frac{S}{a}[1 - e^{-ar}] \ , \tag{2}$$

where $L(0)$ is the initial radiance at distance $r = 0$. In source-free water, $S = 0$ and the solution is a simple exponential decay of the initial radiance with distance:

$$L(r) = L(0)e^{-ar} \ , \tag{3}$$

This result is known as Beer's Law, Lambert's Law, Bouguer's Law, or some hyphenated combination of these names. An equivalent result was derived in another way in the discussion on the Measuring IOPs page; see Eq. (3) of that page.

The exponential decay of light through a medium was first reported in graphical form by Bouguer (1729) (his Fig. 4) based on observations of candles made by eye. Lambert (1760) placed Bouguer's graphical result in the mathematical form an of exponential decay with distance. Beer (1852) found that for a fixed distance, the transmitted light decreased as an exponential function of the concentration of salts in solution. It thus seems that Bouguer should be credited with the original understanding of the exponential decrease of light when traveling through a material medium, and Lambert gets credit for putting Bouguer's results into the mathematical form seen in Eq. (likesection3). It is then "Lambert's law" if you use Eq. (likesection3) to predict the attenuation of light as a function of distance for a given substance, and it is "Beer's law" if you use Eq. (likesection3) to predict the attenuation of light as a function of concentration for a given distance. To give full credit to everyone, it should be called the "Bouguer-Lambert-Beer law." It seems that "Beer's law" or the "Lambert-Beer law" seems to be most common in optical oceanography.

Note that for great distances, $r \to \infty$, the radiance depends only on the source function and the absorption coefficient: $L \to S/a$. This result for the asymptotic behavior of the radiance with distance also holds when scattering is present, as shown on the page for An Analytical Asymptotic Solution for Internal Sources.

Approximate Analytical Solutions

A number of approximate analytical solutions to the SRTE can be derived after simplifying the SRTE in various ways. One of these approximate solutions is the Single-scattering Approximation (SSA). This solution is developed by assuming that the water is homogeneous, the sea surface is level, the sun is a point source in a black sky, there are no internal sources, and only single-scattering of photons is considered. The Quasi-single-scattering Approximation (QSSA) is a further simplification of the SSA. These are discussed in detail in Level 2 and in Gordon (1994). A much more complicated solution including second-order scattering (i.e., including photons that have been scattered twice) is developed in Walker (1994), Section 2-6.

These approximate solutions of the SRTE are useful for isolating the main factors influencing underwater radiances. However, the solutions depend on various simplifying assumptions and the predicted radiances are typically accurate to a few tens of percent at best, and can be off by an order of magnitude.

Numerical Solutions

If accurate solutions of the vector or scalar RTE are to be obtained for realistic oceanic conditions, numerical methods must be used. Many such methods have been developed. Some of these solution techniques have been tailored to specific environments, such as stellar or planetary atmospheres, and are not commonly used in oceanography. The numerical methods most commonly employed in oceanographic radiative transfer, and their salient characteristics, can be summarized as follows:

Monte Carlo

- based on conceptually simple physics that mimics how nature absorbs and scatters idealized light rays
- completely general; can solve time-dependent and 3D problems with arbitrary geometry
- sophisticated mathematical "tricks" can be used to speed up the calculations
- easy to program
- computed radiances have statistical errors, which can be reduced by tracing more initial rays (requiring longer computer times)
- computer run times can be extremely slow for some problems (e.g., solving the RTE to large optical depths; run times increase exponentially with optical depth)

Discrete Ordinates

- highly mathematical
- difficult to program
- does not handle highly peaked scattering phase functions well
- models the medium as a stack of homogeneous layers
- is fast for irradiance calculations and homogeneous water, but can be slow for radiances or if many layers are needed to resolve depth-dependent IOPs

Invariant Imbedding

- highly mathematical
- difficult to program
- can solve only 1D problems (the one dimension being the depth in optical oceanography)
- includes all orders of multiple scattering
- computed radiances do not have statistical errors
- is extremely fast (run times increase linearly with optical depth)

Because of their simplicity and generality, Monte Carlo methods are widely used to solve RTEs in fields as diverse as oceanography, atmospheric sciences, astronomy, medical physics, and nuclear engineering. The trade-off for their simplicity and generality is long computer run times for many problems. Nevertheless, Monte Carlo methods warrant a chapter of their own. The scattering phase functions for atmospheric aerosols are not as highly peaked at very small scattering angles as are those for oceanic particles. Discrete ordinates can handle aerosol phase functions well and is often used in atmospheric optics, but is not much used for underwater calculations because of the need to resolve highly peaked phase functions and to have many layers if the IOPs vary greatly with depth. Invariant imbedding is the solution technique used by the HydroLight numerical model, which is widely used in oceanography. Each of these techniques gives the same answer for the same inputs and boundary conditions for the RTE, for problems where all three techniques are applicable, as can be seen in the model comparison study of Mobley et al. (1993). They differ only in their internal mathematics and the resulting computer run times, and well-debugged and validated computer programs exist for each. In this sense, solving the RTE in the oceanographic setting can be considered a "solved problem."